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THE GAMMA FUNCTION IN THE INTEGRAL CALCULUS.

By T. H. Gronwall.

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Introduction. The object of this paper is to give an exposition, as elementary as possible, of some of those aspects of the theory of the Gamma function which are not dealt with in Jensen's "An elementary exposition of the theory of the Gamma function."[1] Chapter I presents briefly those definitions and theorems on definite integrals which are indispensable in the following. In chapter II, the classical applications of the integral calculus to the Gamma function are set forth in a form which adheres quite closely to the point of view of Jensen's paper, while in chapter III the same body of theorems is derived from the definition of $\Gamma(s)$ as a definite integral.*

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[1] Authorized translation, with additional notes, by T. H. Gronwall. These *Annals*, ser. 2, vol. 17 (1916), pp. 124–166. References to paragraphs and footnotes in this paper will be given thus: J § 2 and J².

While the paper contains little that is new in subject matter, a considerable number of proofs have been remodeled, or replaced by new ones.

CHAPTER I.

Definite Integrals.

1. **Uniform continuity.** Let us consider a function $f(x_1, x_2, \cdots, x_n)$ defined for all values of $x_1, x_2, \cdots, x_n$ belonging to a pointset P which is bounded (i. e., there exists an A such that $|x_1| < A$, $|x_2| < A, \cdots$, $|x_n| < A$ for all points of P). The function is *continuous* on the pointset P when, for any point $x_1, x_2, \cdots, x_n$ of P and any ϵ (arbitrarily small), there exists a $\delta = \delta(\epsilon, x_1, x_2, \cdots, x_n)$ such that for every point $x_1', x_2', \cdots, x_n'$ of P where

$$(1) \qquad |x_1' - x_1| < \delta, \quad |x_2' - x_2| < \delta, \quad \cdots, \quad |x_n' - x_n| < \delta$$

we have

$$(2) \qquad |f(x_1', x_2', \cdots, x_n') - f(x_1, x_2, \cdots, x_n)| < \epsilon.$$

The function is *uniformly continuous on the pointset* P when, for any ϵ, a $\delta = \delta(\epsilon)$ may be chosen *independently* of $x_1, x_2, \cdots, x_n$ such that the inequality (2) holds for any two points $x_1, x_2, \cdots, x_n$ and $x_1', x_2', \cdots, x_n'$ of P satisfying (1). The following theorem is of fundamental importance for the definition of an integral:

THEOREM I. *When $f(x_1, x_2, \cdots, x_n)$ is continuous on a* CLOSED[2] *pointset P, this function is also uniformly continuous on P.*

Suppose the theorem to be false; then there exist an $\epsilon > 0$, an infinite sequence of pairs of points on P: $x_{1\nu}, x_{2\nu}, \cdots, x_{n\nu}$ and $x_{1\nu}', x_{2\nu}', \cdots, x_{n\nu}'$ $(\nu = 1, 2, 3, \cdots)$ such that

$$(3) \qquad |x_{1\nu}' - x_{1\nu}| < \delta_\nu, \quad \cdots, \quad |x_{n\nu}' - x_{n\nu}| < \delta_\nu, \quad \delta_\nu \rightarrow 0 \text{ as } \nu \rightarrow \infty,$$

and for which

$$(4) \qquad |f(x_{1\nu}', x_{2\nu}', \cdots, x_{n\nu}') - f(x_{1\nu}, x_{2\nu}, \cdots, x_{n\nu})| \geqq \epsilon.$$

Let a limiting point of the sequence $x_{1\nu}, x_{2\nu}, \cdots, x_{n\nu}$ be $x_{10}, x_{20}, \cdots, x_{n0}$; this point belongs to P, since P is closed, and $f(x_1, x_2, \cdots, x_n)$ being continuous on P, there exists a δ_0 such that for $|x_1 - x_{10}| < \delta_0$, $|x_2 - x_{20}| < \delta_0, \cdots, |x_n - x_{n0}| < \delta_0$ we have $|f(x_1, x_2, \cdots, x_n) - f(x_{10}, x_{20}, \cdots, x_{n0})| < \epsilon/2$. We may now choose ν so large that $\delta_\nu < \delta_0/2$ and $|x_{1_\nu} - x_{10}| < \delta_0/2, \cdots, |x_{n_\nu} - x_{n0}| < \delta_0/2$; hence, by (3), $|x_{1_\nu}' - x_{10}| \leqq |x_{1_\nu} - x_{10}| + |x_{1_\nu}' - x_{1_\nu}| < \delta_0/2 + \delta_\nu < \delta_0, \cdots, |x_{n_\nu}' - x_{n0}| < \delta_0$. We have therefore $|f(x_{1_\nu}, x_{2_\nu}, \cdots, x_{n_\nu}) - f(x_{10}, x_{20}, \cdots, x_{n0}| < \epsilon/2$, $|f(x_{1_\nu}', x_{2_\nu}', \cdots, x_{n_\nu}') - f(x_{10}, x_{20}, \cdots, x_{n0})| < \epsilon/2$, whence by addition $|f(x_{1_\nu}', x_{2_\nu}', \cdots, x_{n_\nu}') - f(x_{1_\nu}, x_{2_\nu}, \cdots, x_{n_\nu})| < \epsilon$, which contradicts (4), and this contradiction proves our theorem.

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* See the note at the end of this paper.

[2] I. e., a pointset containing all its limiting points.

2. **Upper and lower integrals of a function.** We define the interval (a, b), where $a < b$, as the set of all points x such that $a \leqq x \leqq b$. This pointset is evidently closed; a and b are called the end points, and the points x such that $a < x < b$ the interior points of the interval. The *length* of the interval is $b - a$.

A set of values of a real variable ξ is said to be bounded *below*, when there exists a finite A such that $\xi > A$ for every ξ in the set, and bounded *above*, when $\xi < B$ for every ξ. It is a well-known theorem that every set of ξ's which is bounded below has a *lower bound* m such that $\xi \geqq m$ for every ξ but that for any positive ϵ there exists at least one ξ in the set such that $\xi < m + \epsilon$; similarly every set which is bounded above has an *upper bound* M such that $\xi \leqq M$ for every ξ but $\xi > M - \epsilon$ for at least one ξ in the set. It is evident that the lower and upper bounds of $-\xi$ are $-M$ and $-m$ respectively.

In particular, let ξ be the set of values assumed by a function $f(x)$ when x varies in the interval (a, b); when this set of ξ's is bounded both above and below, $f(x)$ is said to be *bounded* in the interval (a, b), and m and M are called the lower and upper bounds of $f(x)$ in this interval.

Now consider a function $f(x)$, bounded in the interval (a, b), and let us divide this interval into a number of subintervals $(x_{\nu-1}, x_\nu)$ (of length $x_\nu - x_{\nu-1}$) by means of the points of division $x_0, x_1, \cdots, x_n$, where $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Denote by M_ν the upper, and by m_ν the lower bound of $f(x)$ in the interval $(x_{\nu-1}, x_\nu)$, and write

$$S = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1}) = \Sigma M_\nu(x_\nu - x_{\nu-1}),$$

$$s = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1}) = \Sigma m_\nu(x_\nu - x_{\nu-1}).$$

Evidently $M \geqq M_\nu \geqq m_\nu \geqq m$, and hence

$$M(b - a) \geqq S \geqq s \geqq m(b - a). \tag{5}$$

If we consider all possible values of S and s corresponding to all possible modes of subdivision of (a, b), S and s are bounded above and below by (5); hence there exist an upper bound j of s and a lower bound J of S. We shall now prove

THEOREM II. *When the number of subintervals is increased indefinitely in such a manner that the length of each tends toward zero, the sums S and s tend toward the limits J and j respectively.*

Let us change the sum s into s' by introducing a new division point x' between $x_{\nu-1}$ and x_ν. Denoting by m_ν' and m_ν'' the lower bounds of $f(x)$ in $(x_{\nu-1}, x')$ and (x', x_ν), we obviously have

$$\begin{aligned} s' - s &= m_\nu'(x' - x_{\nu-1}) + m_\nu''(x_\nu - x') - m_\nu(x_\nu - x_{\nu-1}) \\ &= (m_\nu' - m_\nu)(x' - x_{\nu-1}) + (m_\nu'' - m_\nu)(x_\nu - x'), \end{aligned}$$

and since $0 \leqq m_\nu' - m_\nu \leqq M - m$, $0 \leqq m_\nu'' - m_\nu \leqq M - m$, we find $0 \leqq s' - s \leqq (M - m)(x_\nu - x_{\nu-1}) < (M - m)\delta$ if $x_\nu - x_{\nu-1} < \delta$. More generally, if s' is obtained from s by the introduction of k new division points, we find by introducing them one at a time and successive application of the above inequality that

$$0 \leqq s' - s < k(M - m)\delta, \tag{6}$$

if every subinterval corresponding to s is less than δ. Since $-J$ and $-S$ stand in the same relation to $-f(x)$ as j and s to $f(x)$, it is sufficient to show that for any $\epsilon > 0$ there exists a δ such that $j \geqq s > j - \epsilon$ for any s in which the length of each subinterval is less than δ. Since j is the upper bound of all s, it is clear that $j \geqq s$ and that there exists an s_0 such that $s_0 > j - \epsilon/2$. Let the number of subdivisions corresponding to s_0 be k, and consider any s in which the length of each subdivision is less than

$$\delta = \frac{\epsilon}{2k(M - m)}.$$

Finally let s' correspond to the subdivision having as division points all those occurring in s and in s_0; then, by (6),

$$s > s' - k(M - m)\delta = s' - \epsilon/2,$$

and applying (6) to s_0 and s', we find $s' \geqq s_0$, so that

$$s > s_0 - \epsilon/2 > (j - \epsilon/2) - \epsilon/2 = j - \epsilon,$$

which proves our theorem.

The numbers J and j are called the upper and lower integrals of $f(x)$ between the limits a and b, and are denoted by

$$J = \overline{\int_a^b} f(x)dx, \qquad j = \underline{\int_a^b} f(x)dx.$$

Since $S \geqq s$, it follows from theorem II that $J \geqq j$ or

$$\overline{\int_a^b} f(x)dx \geqq \underline{\int_a^b} f(x)dx. \tag{7}$$

Let $a < c < b$ and consider the sums S and s for which c is a division point, it follows immediately from theorem II that

$$\begin{aligned} &\overline{\int_a^b} f(x)dx = \overline{\int_a^c} f(x)dx + \overline{\int_c^b} f(x)dx, \\ &\underline{\int_a^b} f(x)dx = \underline{\int_a^c} f(x)dx + \underline{\int_c^b} f(x)dx, \end{aligned} \tag{8}$$

and from (5) and (7) it is seen that

$$m(b - a) \leqq \underline{\int_a^b} f(x)dx \leqq \overline{\int_a^b} f(x)dx \leqq M(b - a). \tag{9}$$

Furthermore, when k is a constant, the sums S and s for $kf(x)$ equal k times those for $f(x)$, whence

$$\begin{aligned} &\overline{\int_a^b} kf(x)dx = k\overline{\int_a^b} f(x)dx, \qquad \underline{\int_a^b} kf(x)dx = k\underline{\int_a^b} f(x)dx \text{ for } k \geqq 0, \\ &\overline{\int_a^b} kf(x)dx = k\underline{\int_a^b} f(x)dx, \qquad \underline{\int_a^b} kf(x)dx = k\overline{\int_a^b} f(x)dx \text{ for } k \leqq 0. \end{aligned} \tag{10}$$

Finally, let $f_1(x), f_2(x), \cdots, f_n(x)$ all be bounded in (a, b), and let M_ν and m_ν, $M_{1\nu}$ and $m_{1\nu}$, $\cdots$, $M_{n\nu}$ and $m_{n\nu}$ be the upper and lower bounds of $(f_1(x) + f_2(x) + \cdots + f_n(x))$, $f_1(x)$, $\cdots$, $f_n(x)$ in $(x_{\nu-1}, x_\nu)$. Evidently $M_\nu \leqq M_{1\nu} + M_{2\nu} + \cdots + M_{n\nu}$ and $m_\nu \geqq m_{1\nu} + m_{2\nu} + \cdots + m_{n\nu}$, whence

$$\Sigma M_\nu(x_\nu - x_{\nu-1}) \leqq \Sigma M_{1\nu}(x_\nu - x_{\nu-1}) + \cdots + \Sigma M_{n\nu}(x_\nu - x_{\nu-1})$$

and

$$\Sigma m_\nu(x_\nu - x_{\nu-1}) \geqq \Sigma m_{1\nu}(x_\nu - x_{\nu-1}) + \cdots + \Sigma m_{n\nu}(x_\nu - x_{\nu-1}).$$

Passing to the limit, we obtain

$$\begin{aligned} &\overline{\int_a^b} (f_1(x) + f_2(x) + \cdots + f_n(x))dx \\ &\qquad \leqq \overline{\int_a^b} f_1(x)dx + \overline{\int_a^b} f_2(x)dx + \cdots + \overline{\int_a^b} f_n(x)dx, \\ &\underline{\int_a^b} (f_1(x) + f_2(x) + \cdots + f_n(x))dx \\ &\qquad \geqq \underline{\int_a^b} f_1(x)dx + \underline{\int_a^b} f_2(x)dx + \cdots + \underline{\int_a^b} f_n(x)dx. \end{aligned} \tag{11}$$

3. **The definite integral.** When the function $f(x)$ is bounded in the interval (a, b) and the upper and lower integrals of $f(x)$ between the limits a and b are equal, the function is said to be *integrable* in (a, b), and the common value of the upper and lower integrals is called the integral

of $f(x)$ between the limits a and b and is denoted by

$$\int_a^b f(x)dx.$$

The necessary and sufficient condition for integrability is obviously that the difference (which is $\geqq 0$ by (5))

$$S - s = \Sigma(M_\nu - m_\nu)(x_\nu - x_{\nu-1})$$

shall tend toward zero for any one particular mode of subdivision of (a, b) when the number of subdivisions is increased indefinitely in such a manner that the length of each approaches zero.

We shall now prove

THEOREM III. *A bounded function $f(x)$ is integrable in the interval (a, b) when it is possible to find a finite number of subintervals, the total length of which is as small as we please, and such that every point of discontinuity of $f(x)$ is interior to one of these subintervals.*

We may note that the hypothesis of this theorem is fulfilled when $f(x)$ has a finite number of points of discontinuity in (a, b).

To prove our theorem, it suffices, by the remark above, to exhibit a subdivision such that the corresponding sums S and s differ by less than any assigned ϵ. By hypothesis, we can find a finite number of intervals Δ', of total length less than $\frac{\epsilon}{2(M - m)}$, such that every point of discontinuity of $f(x)$ is interior to one of them. Denote by Δ'' the finite number of intervals obtained by removing from (a, b) the *interior* points of all intervals Δ', and consider a subdivision of (a, b) in which the end points of all intervals Δ' are among the division points. Then we may write

$$S - s = \Sigma'(M_\nu - m_\nu)(x_\nu - x_{\nu-1}) + \Sigma''(M_\nu - m_\nu)(x_\nu - x_{\nu-1}),$$

where Σ' and Σ'' refer to the intervals belonging to Δ' and Δ'' respectively. For the former, we have, since $M_\nu - m_\nu \leqq M - m$,

$$\Sigma'(M_\nu - m_\nu)(x_\nu - x_{\nu-1})$$
$$\leqq (M - m)\Sigma'(x_\nu - x_{\nu-1}) < (M - m) \cdot \frac{\epsilon}{2(M - m)} = \frac{\epsilon}{2}.$$

Regarding Σ'', we observe that Δ'' is a closed pointset on which $f(x)$ is continuous, and by theorem I we may assign a δ so small that

$$|f(x') - f(x'')| < \frac{\epsilon}{2(b - a)}$$

for any x', x'' both belonging to Δ'' and such that $|x' - x''| < \delta$. Now make every subdivision in Δ'' so small that $x_\nu - x_{\nu-1} < \delta$, and let x', x'' be values of x in the interval $(x_{\nu-1}, x_\nu)$ for which the continuous function $f(x)$ takes its maximum and minimum values M_ν and m_ν respectively. Then

$$M_\nu - m_\nu < \frac{\epsilon}{2(b-a)}$$

for every subinterval belonging to Δ'', and the total length of these subintervals not exceeding $b - a$, we have

$$\Sigma'' < \frac{\epsilon}{2(b-a)} \cdot (b-a) = \frac{\epsilon}{2}.$$

Consequently

$$0 \leqq S - s < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which proves our proposition.

THEOREM IV. *When $f(x)$ is bounded and integrable in (a, b), the same is true of $|f(x)|$.*

It is evident that $|f(x)|$ is bounded when $f(x)$ is bounded. Let M_ν and m_ν, M_ν' and m_ν' be the upper and lower bounds of $f(x)$ and $|f(x)|$ in $(x_{\nu-1}, x_\nu)$. When $M_\nu \geqq m_\nu \geqq 0$, we have $M_\nu' = M_\nu$, $m_\nu' = m_\nu$; when $0 \geqq M_\nu \geqq m_\nu$, then $M_\nu' = -m_\nu$, $m_\nu' = -M_\nu$, and when $M_\nu \geqq 0 \geqq m_\nu$, M_ν' equals the greater of M_ν and $-m_\nu$, while $m_\nu' = 0$, so that in all three cases $0 \leqq M_\nu' - m_\nu' \leqq M_\nu - m_\nu$. Since $\Sigma(M_\nu - m_\nu)(x_\nu - x_{\nu-1})$ tends toward zero, the same is true of $\Sigma(M_\nu' - m_\nu')(x_\nu - x_{\nu-1})$, so that the upper and lower integrals of $|f(x)|$ are equal.

From (9) it follows that when $f(x)$ is integrable in (a, b)

$$m(b-a) \leqq \int_a^b f(x)dx \leqq M(b-a). \tag{12}$$

More generally, when three bounded and integrable functions $f(x)$, $m(x)$ and $M(x)$ satisfy the inequalities $m(x) \leqq f(x) \leqq M(x)$ in (a, b), we have

$$\int_a^b m(x)dx \leqq \int_a^b f(x)dx \leqq \int_a^b M(x)dx. \tag{13}$$

In fact, since $m(x) \leqq f(x) \leqq M(x)$ in $(x_{\nu-1}, x_\nu)$, the lower bound of $m(x)$ in this interval does not exceed that of $f(x)$, and the upper bound of $f(x)$ does not exceed that of $M(x)$; hence the sum s relative to $m(x)$ does not exceed that relative to $f(x)$, and the S relative to $f(x)$ does not exceed that relative to $M(x)$. Passing to the limit, we obtain (13).

In particular, theorem IV allows us to take $m(x) = -|f(x)|$,

$M(x) = |f(x)|$, and (13) gives

$$\left|\int_a^b f(x)dx\right| \leqq \int_a^b |f(x)|\,dx. \tag{14}$$

When $a < c < b$, (8) becomes for an integrable $f(x)$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

and the repeated application of this formula gives

$$\begin{aligned}\int_a^b f(x)dx = \int_a^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx \\ + \cdots + \int_{x_{n-2}}^{x_{n-1}} f(x)dx + \int_{x_{n-1}}^b f(x)dx\end{aligned} \tag{15}$$

for $a < x_1 < x_2 < \cdots < x_{n-1} < b$.

It is sometimes convenient to consider the case where the upper limit in an integral is smaller than the lower one; in this case we introduce the definition

$$\int_b^a f(x)dx = -\int_a^b f(x)dx, \qquad a < b. \tag{16}$$

From (16) it follows that (15) is valid when $a, x_1, x_2, \cdots, x_{n-1}, b$ are in any order of magnitude, provided $f(x)$ is integrable in an interval containing them all. When $b < a$, the inequalities (12), (13), (14) must however be reversed.

The inequalities (10) and (11) give when $f(x), f_1(x), \cdots, f_n(x)$ are integrable

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx, \tag{17}$$

$$\begin{aligned}\int_a^b (f_1(x) + f_2(x) + \cdots + f_n(x))dx \\ = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx + \cdots + \int_a^b f_n(x)dx.\end{aligned} \tag{18}$$

Theorem V. *When $f_1(x)$ and $f_2(x)$ are bounded and integrable in (a, b), the same is true of their product $f(x) = f_1(x)f_2(x)$.*

Let M and m, M' and m', M'' and m'' be the upper and lower bounds of $f(x), f_1(x)$ and $f_2(x)$ in (a, b); furthermore, let M_ν and m_ν, M_ν' and m_ν', M_ν'' and m_ν'' be the corresponding bounds in $(x_{\nu-1}, x_\nu)$. Evidently $|f(x)|$ does not exceed the greatest of the quantities $M'M''$, $m'm''$, $-M'm''$, $-m'M''$, so that $f(x)$ is bounded in (a, b). To prove the integrability, let us first suppose that $f_1(x)$ and $f_2(x)$ (and consequently $f(x)$) are never negative in (a, b). Then obviously $M_\nu \leqq M_\nu' M_\nu''$, $m_\nu \geqq m_\nu' m_\nu''$, and

$$M_\nu - m_\nu \leqq M_\nu' M_\nu'' - m_\nu' m_\nu''$$
$$= M_\nu''(M_\nu' - m_\nu') + m_\nu'(M_\nu'' - m_\nu'') \leqq M''(M_\nu' - m_\nu') + M'(M_\nu'' - m_\nu'');$$

hence

$$\Sigma(M_\nu - m_\nu)(x_\nu - x_{\nu-1}) \leqq M''\Sigma(M_\nu' - m_\nu')(x_\nu - x_{\nu-1}) + M'\Sigma(M_\nu'' - m_\nu'')(x_\nu - x_{\nu-1}).$$

$f_1(x)$ and $f_2(x)$ being integrable, both sums to the right may be made as small as we please by taking the subdivisions sufficiently small, and consequently the same is true of the sum to the left, i. e., $f(x)$ is integrable. Second, if $f_1(x)$ or $f_2(x)$ assumes negative values in (a, b), the product of the non-negative functions $f_1(x) - m'$ and $f_2(x) - m''$ is integrable, and by (17) and (18), the same is the case for

$$f_1(x)f_2(x) = (f_1(x) - m')(f_2(x) - m'') + m''f_1(x) + m'f_2(x) - m'm''.$$

THEOREM VI. *When $f(x)$ is integrable in (a, b), and for any x between a and b we write*

$$F(x) = \int_a^x f(t)dt, \qquad x \neq a; \qquad F(a) = 0,$$

then $F(x)$ is continuous in (a, b), and at any point where $f(x)$ is continuous, we have

$$\frac{dF(x)}{dx} = f(x).$$

By (15) we have, when both x and $x + h$ are in (a, b),

$$F(x + h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt,$$

and if M and m are the upper and lower bounds of $f(t)$ in $(x, x + h)$, (12) gives

$$mh \leqq F(x + h) - F(x) \leqq Mh$$

(if h is negative, the inequalities must be reversed by (16)), which proves the continuity of $F(x)$. When $f(x)$ is continuous at x, m and M tend toward the same limit $f(x)$ as $h \to 0$, and consequently

$$f(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = \frac{dF(x)}{dx}.$$

THEOREM VII. *When $f(x)$ satisfies the conditions of theorem III, and* *there exists a function* $F_1(x)$ *continuous in* (a, b) *and the derivative of which equals* $f(x)$ *at every point where* $f(x)$ *is continuous, then*

$$F_1(x) - F_1(a) = \int_a^x f(t)dt$$

for any x *in* (a, b).

Let $F(x)$ be the function defined in theorem VI; then $F_1(x)$ and $F(x)$ have the same derivative $f(x)$ in any subinterval of (a, b) in which $f(x)$ is continuous, and by the theorem of the mean in the differential calculus, $F_1(x) = F(x) +$ const. in the subinterval considered. Now let (x_1, x_2) and (x_3, x_4) be two subintervals of continuity for $f(x)$, separated by the subinterval (x_2, x_3) in the interior of which $f(x)$ becomes discontinuous. Then we have $F_1(x) = F(x) + C$ in (x_1, x_2) and $F_1(x) = F(x) + C'$ in (x_3, x_4), and consequently

$$C' - C = (F_1(x_3) - F_1(x_2)) - (F(x_3) - F(x_2)).$$

By the conditions of theorem III, $x_3 - x_2$ may be made as small as we please; since $F_1(x)$ is continuous in (a, b) by hypothesis and $F(x)$ by theorem VI, the expression to the right may be made as small as we please by taking $x_3 - x_2$ sufficiently small, and $C' - C$, being independent of x_2 and x_3, therefore equals zero. Consequently

$$F_1(x) = F(x) + C$$

throughout the interval (a, b), and since $F(a) = 0$, we have $C = F_1(a)$ and the theorem is proved.

THEOREM VIII. *When* $f(x)$, $f_1(x)$ *and* $f_2(x)$ *are bounded and integrable in* (a, b), *then*

$$\int_a^b f(x)dx = \lim_{\delta \to 0} \Sigma f(\xi_\nu)(x_\nu - x_{\nu-1}), \tag{19}$$

$$\int_a^b f_1(x)f_2(x)dx = \lim_{\delta \to 0} \Sigma f_1(\xi_\nu')f_2(\xi_\nu'')(x_\nu - x_{\nu-1}), \tag{20}$$

where ξ_ν, ξ_ν' *and* ξ_ν'' *are any points in* $(x_{\nu-1}, x_\nu)$, *and* $x_\nu - x_{\nu-1} < \delta$ *for all* ν. *When* ξ_ν *is a point of discontinuity of* $f(x)$, *we may take as the value of* $f(\xi_\nu)$ *any number between* m *and* M.

In (19), the sum to the right lies between s and S, and consequently tends toward the same limit $\int_a^b f(x)dx$ as these. In (20), we have

$$\begin{aligned}\Sigma f_1(\xi_\nu')f_2(\xi_\nu'')(x_\nu - x_{\nu-1})\\ = \Sigma f_1(\xi_\nu')f_2(\xi_\nu')(x_\nu - x_{\nu-1}) + \Sigma f_1(\xi_\nu')(f_2(\xi_\nu'') - f_2(\xi_\nu'))(x_\nu - x_{\nu-1});\end{aligned}$$

the first sum to the right tends toward $\int_a^b f_1(x)f_2(x)dx$ as $\delta \to 0$ by theorem

V and (19), while the absolute value of the second sum does not exceed

$$\begin{aligned}\Sigma\,|f_1(\xi_\nu')| \cdot |f_2(\xi_\nu'') - f_2(\xi_\nu')|\,(x_\nu - x_{\nu-1}) \\ \leqq \Sigma\,|f_1(\xi_\nu')|\,(M_\nu'' - m_\nu'')(x_\nu - x_{\nu-1}) < (\,|M'| + |m'|\,)\Sigma(M_\nu'' \\ - m_\nu'')(x_\nu - x_{\nu-1})\end{aligned}$$

and $f_2(x)$ being integrable, the last sum approaches zero as $\delta \rightarrow 0$.

THEOREM IX. **Integration by parts.** *Let $u(x)$ and $v(x)$ be continuous functions of x such that bounded functions $u'(x)$ and $v'(x)$ exist in (a, b) and are continuous and equal to the derivatives of $u(x)$ and $v(x)$ except possibly on a pointset which may be enclosed in the interior of a finite number of subintervals of arbitrarily small total length; then*

$$\int_a^b u(x)v'(x)dx = u(b)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)dx. \tag{21}$$

Except on the pointset specified, we have

$$uv' + vu' = \frac{d(uv)}{dx},$$

and the integral $\int_a^b (uv' + vu')dx$, which exists by theorem V and (18), equals $u(b)v(b) - u(a)v(a)$ by theorem VII, whence (21) follows by applying (18).

THEOREM X. **Integration by substitution.** *Let $f(x)$ be bounded and integrable in (a, b), and $\varphi(t)$ a function such that $\varphi(\alpha) = a$, $\varphi(\beta) = b$, and $\varphi'(t)$ is continuous and different from zero in (α, β); then*

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt. \tag{22}$$

Supposing that $\varphi'(t) > 0$ in (α, β), there exists a $k > 0$ such that $\varphi'(t) \geqq k$ in (α, β). Subdivide (α, β) at the points $\alpha = t_0, t_1, \cdots, t_{n-1}, t_n = \beta$, and write $x_\nu = \varphi(t_\nu)$. Let M_ν and m_ν be the upper and lower bounds of $f(\varphi(t))$ in $(t_{\nu-1}, t_\nu)$ or, which is the same, the upper and lower bounds of $f(x)$ in $(x_{\nu-1}, x_\nu)$. Then, by the mean value theorem,

$$x_\nu - x_{\nu-1} = \varphi'(t_\nu')(t_\nu - t_{\nu-1}) > k(t_\nu - t_{\nu-1}) > 0,$$

where t' is some value in $(t_{\nu-1}, t_\nu)$, so that

$$0 < \Sigma(M_\nu - m_\nu)(t_\nu - t_{\nu-1}) < \frac{1}{k}\Sigma(M_\nu - m_\nu)(x_\nu - x_{\nu-1}),$$

and $f(x)$ being integrable in (a, b), the sum to the right approaches zero when all $x_\nu - x_{\nu-1}$ tend toward zero, which is evidently the case when all

$t_\nu - t_{\nu-1}$ tend toward zero, and the above inequality shows that $f(\varphi(t))$ is integrable in (α, β). The same is true when $\varphi'(t) \leqq -k$ in (α, β), the above inequality being then reversed. Since $\varphi'(t)$ is integrable, the product $f(\varphi(t))\varphi'(t)$ is integrable by theorem V, and for t_ν'' in $(t_{\nu-1}, t_\nu)$, $x_\nu' = \varphi(t_\nu'')$, we have

$$\Sigma f(x_\nu')(x_\nu - x_{\nu-1}) = \Sigma f(\varphi(t_\nu''))\varphi'(t_\nu')(t_\nu - t_{\nu-1}).$$

Passing to the limit and using theorem VIII, our proposition follows.

4. **Double and repeated integrals.** Let $f(x, y)$ be bounded in the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, and subdivide this rectangle into the mn rectangles $x_{\mu-1} \leqq x \leqq x_\mu$, $y_{\nu-1} \leqq y \leqq y_\nu$, where

$$a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$$

and

$$c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d.$$

Denote by M and m the upper and lower bounds of $f(x, y)$ in the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, and by $M_{\mu\nu}$ and $m_{\mu\nu}$ the corresponding bounds in the rectangle $x_{\mu-1} \leqq x \leqq x_\mu$, $y_{\nu-1} \leqq y \leqq y_\nu$, and consider the two sums

$$S = \sum_{\mu=1}^{m}\sum_{\nu=1}^{n} M_{\mu\nu}(x_\mu - x_{\mu-1})(y_\nu - y_{\nu-1}),$$

$$s = \sum_{\mu=1}^{m}\sum_{\nu=1}^{n} m_{\mu\nu}(x_\mu - x_{\mu-1})(y_\nu - y_{\nu-1}).$$

Since evidently $M \geqq M_{\mu\nu} \geqq m_{\mu\nu} \geqq m$, we have

$$M(b - a)(d - c) \geqq S \geqq s \geqq m(b - a)(d - c), \tag{23}$$

so that for all possible modes of subdivision of the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, S and s are bounded above and below; hence there exist an upper bound j of s and a lower bound J of S. In exactly the same way as we proved theorem II, we find

THEOREM XI. *When the number of subrectangles is increased indefinitely in such a manner that the sides $x_\mu - x_{\mu-1}$ and $y_\nu - y_{\nu-1}$ of each tend toward zero, the sums S and s tend toward the limits J and j respectively.* The numbers J and j are called the upper and lower integrals of $f(x, y)$ over the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, and are denoted by

$$J = \overline{\int_a^b \int_c^d} f(x, y)\,dxdy, \qquad j = \underline{\int_a^b \int_c^d} f(x, y)\,dxdy.$$

When $f(x, y)$ is bounded in the rectangle, and the upper and lower integrals of $f(x, y)$ over the rectangle are equal, the function is said to be *integrable* in the rectangle, and the common value of J and j is called the (double) integral of $f(x, y)$ and denoted by

$$\int_a^b \int_c^d f(x, y)\,dx\,dy.$$

The necessary and sufficient condition for integrability is obviously that the difference (which is $\geqq 0$ by (23))

$$S - s = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} (M_{\mu\nu} - m_{\mu\nu})(x_\mu - x_{\mu-1})(y_\nu - y_{\nu-1})$$

shall tend toward zero for some particular mode of subdivision of the rectangle when the number of subrectangles is increased indefinitely in such a manner that the lengths of all their sides approach zero.

The following theorem is proved in exactly the same way as theorem III:

THEOREM XII. *A bounded function $f(x, y)$ is integrable in the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, when it is possible to find a finite number of subrectangles, the total area of which is as small as we please, and such that every point of discontinuity of $f(x, y)$ is interior to one of these subrectangles.*

For any value of x in (a, b), $f(x, y)$ is bounded when $c \leqq y \leqq d$, and consequently the upper and lower integrals $\overline{\int_c^d} f(x, y)\,dy$ and $\underline{\int_c^d} f(x, y)\,dy$ exist. Denote by

$$F(x) = \overline{\underline{\int_c^d}} f(x, y)\,dy$$

any number such that

$$\overline{\int_c^d} f(x, y)\,dy \geqq \overline{\underline{\int_c^d}} f(x, y)\,dy \geqq \underline{\int_c^d} f(x, y)\,dy,$$

so that, in particular, $F(x) = \int_c^d f(x, y)\,dy$ for every value of x for which $f(x, y)$ is integrable in respect to y between the limits c and d. Then the integral (which we suppose for the moment to exist) $\int_a^b F(x)\,dx$ is called the *repeated integral* of $f(x, y)$ in respect to y and x and is denoted by

$$\int_a^b dx \int_c^d f(x, y)\,dy.$$

Similarly, we may define the repeated integral in respect to x and y,

$$\int_c^d dy \int_a^b f(x, y)\,dx = \int_c^d G(y)\,dy,$$

where $G(y) = \overline{\underline{\int_a^b}} f(x, y)\,dx$. We then have

THEOREM XIII. *When $f(x, y)$ is bounded and integrable in the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, then both repeated integrals of $f(x, y)$ exist and are equal to the double integral:*

$$\int_a^b dx \int_c^d f(x, y)dy = \int_c^d dy \int_a^b f(x, y)dx = \int_a^b \int_c^d f(x, y)dxdy.$$

Since $m_{\mu\nu} \leqq f(x, y) \leqq M_{\mu\nu}$ for $x_{\mu-1} \leqq x \leqq x_\mu$, $y_{\nu-1} \leqq y \leqq y_\nu$, it follows from (9) that

$$m_{\mu\nu}(y_\nu - y_{\nu-1}) \leqq \underline{\int}_{y_{\nu-1}}^{y_\nu} f(x, y)dy \leqq \overline{\int}_{y_{\nu-1}}^{y_\nu} f(x, y)dy \leqq M_{\mu\nu}(y_\nu - y_{\nu-1});$$

adding these inequalities for $\nu = 1, 2, \cdots, n$ and applying (8), we find

$$\sum_{\nu=1}^{n} m_{\mu\nu}(y_\nu - y_{\nu-1}) \leqq \underline{\int}_c^d f(x, y)dy \leqq \overline{\int}_c^d f(x, y)dy \leqq \sum_{\nu=1}^{n} M_{\mu\nu}(y_\nu - y_{\nu-1})$$

for any x in $(x_{\mu-1}, x_\mu)$, and consequently the lower bound m_μ and the upper bound M_μ of $F(x) = \overline{\underline{\int}}_c^d f(x, y)dy$ for $x_{\mu-1} \leqq x \leqq x_\mu$ satisfy the inequalities

$$\sum_{\nu=1}^{n} m_{\mu\nu}(y_\nu - y_{\nu-1}) \leqq m_\mu \leqq M_\mu \leqq \sum_{\nu=1}^{n} M_{\mu\nu}(y_\nu - y_{\nu-1}).$$

Multiplication by $x_\mu - x_{\mu-1}$ and summation in respect to μ now gives

$$\begin{aligned}\sum_{\mu=1}^{m}\sum_{\nu=1}^{n} m_{\mu\nu}(x_\mu - x_{\mu-1})(y_\nu - y_{\nu-1}) &\leqq \sum_{\mu=1}^{m} m_\mu(x_\mu - x_{\mu-1}) \\ &\leqq \sum_{\mu=1}^{m} M_\mu(x_\mu - x_{\mu-1}) \\ &\leqq \sum_{\mu=1}^{m}\sum_{\nu=1}^{n} M_{\mu\nu}(x_\mu - x_{\mu-1})(y_\nu - y_{\nu-1}).\end{aligned}$$

But as m and n tend toward infinity while all $x_\mu - x_{\mu-1}$ and $y_\nu - y_{\nu-1}$ tend toward zero, the two outside members of this inequality approach the same limit $\int_a^b \int_c^d f(x, y)dxdy$; consequently the two inner members must also approach this same limit, that is, $\int_a^b F(x)dx$ exists and equals the double integral. The same proof obviously applies also to the other repeated integral.

THEOREM XIV. *When $f(x, y)$ and $\dfrac{\partial f(x, y)}{\partial y}$ are bounded for x in (a, b) and y in (c, d) and*

$$\int_a^b \int_c^y \frac{\partial f(x, t)}{\partial t} dxdt$$

exists for any y in (c, d),

$$\int_c^y \frac{\partial f(x, t)}{\partial t} dt$$

exists for any x in (a, b) and any y in (c, d), and finally $\int_a^b f(x, c)dx$ exists, then

$$\frac{\partial}{\partial y}\int_a^b f(x, y)dx = \int_a^b \frac{\partial f(x, y)}{\partial y} dx$$

for every value of y for which the last integral is a continuous function of y.

It may be noted that all the conditions of the theorem are satisfied when $f(x, y)$ and $\partial f(x, y)/\partial y$ are continuous for x in (a, b) and y in (c, d).

For the proof we observe that by theorem XIII

$$(24) \quad \int_a^b \int_c^y \frac{\partial f(x, t)}{\partial t} dxdt = \int_a^b dx \int_c^y \frac{\partial f(x, t)}{\partial t} dt = \int_c^y dt \int_a^b \frac{\partial f(x, t)}{\partial t} dx,$$

and by theorem VII

$$\int_c^y \frac{\partial f(x, t)}{\partial t} dt = f(x, y) - f(x, c);$$

hence, by theorem XIII, $\int_a^b (f(x, y) - f(x, c))dx$ exists, and from the existence of $\int_a^b f(x, c)dx$ and (18) we conclude that

$$\int_a^b f(x, y)dx = \int_a^b f(x, c)dx + \int_a^b (f(x, y) - f(x, c))dx.$$

Consequently (24) gives

$$\int_a^b f(x, y)dx - \int_a^b f(x, c)dx = \int_c^y dt \int_a^b \frac{\partial f(x, t)}{\partial t} dx,$$

and differentiating in respect to y, theorem VI gives

$$\frac{\partial}{\partial y}\int_a^b f(x, y)dx = \int_a^b \frac{\partial f(x, y)}{\partial y} dx$$

for any value of y for which the integral to the right is a continuous function of y.

The following proposition is proved in exactly the same way as theorem IV:

THEOREM XV. *When $f(x, y)$ is bounded and integrable in the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$, the same is true of $|f(x, y)|$.*

Formulas (13) and (14) are also immediately extended to double integrals:

$$(25)\quad \int_a^b\int_c^d m(x, y)dxdy \leqq \int_a^b\int_c^d f(x, y)dxdy \leqq \int_a^b\int_c^d M(x, y)dxdy$$

when the three bounded and integrable functions $f(x, y)$, $m(x, y)$ and $M(x, y)$ satisfy the inequalities $m(x, y) \leqq f(x, y) \leqq M(x, y)$ in the rectangle of integration; and for $m(x, y) = -|f(x, y)|$, $M(x, y) = |f(x, y)|$,

$$(26)\quad \left|\int_a^b\int_c^d f(x, y)dxdy\right| \leqq \int_a^b\int_c^d |f(x, y)|\, dxdy.$$

Also (17) and (18) extend immediately to double integrals:

$$(27)\quad \int_a^b\int_c^d kf(x, y)dxdy = k\int_a^b\int_c^d f(x, y)dxdy,$$

$$\begin{aligned}&\int_b^b\int_d^d (f_1(x, y) + f_2(x, y) + \cdots + f_n(x, y))dxdy\\ (28)\quad &= \int_a^b\int_c^d f_1(x, y)dxdy + \int_a^b\int_c^d f_2(x, y)dxdy + \cdots\\ &\qquad + \int_a^b\int_c^d f_n(x, y)dxdy.\end{aligned}$$

In (25)–(28), any double integral may evidently be replaced by either of the repeated integrals. Finally, theorem VII extends immediately to double and repeated integrals.

5. **Infinite integrals.** In the preceding paragraphs, we have assumed a and b finite and $f(x)$ bounded in (a, b). When one of these conditions is not fulfilled, we may define, by a limiting process, what is called an infinite integral (the term is chosen in analogy to "infinite series").

First assume $f(x)$ to be bounded and integrable in (a, b) for any $b > a$. The integral between the limits a and ∞ is then defined as

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx,$$

whenever this limit exists. Similarly we may define

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx, \quad \int_{-\infty}^\infty f(x)dx = \lim_{\substack{a\to-\infty\\ b\to\infty}}\int_a^b f(x)dx.$$

Next let us assume that $f(x)$ is bounded and integrable in $(a, b-\epsilon)$ for an arbitrarily small positive ϵ, but tends toward $+\infty$ or $-\infty$ (or oscillates between these bounds) as $x \to b$. The integral between the limits a and b is then defined as

$$\int_a^b f(x)dx = \lim_{\epsilon\to 0}\int_a^{b-\epsilon} f(x)dx$$

whenever this limit exists. Further, when $f(x)$ ceases to be bounded as $x \to a$, we define

$$\int_a^b f(x)dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b f(x)dx,$$

and when c is a point between a and b in the neighborhood of which $f(x)$ is not bounded

$$\int_a^b f(x)dx = \lim_{\epsilon \to 0} \int_a^{c-\epsilon} f(x)dx + \lim_{\epsilon' \to 0} \int_{c+\epsilon'}^b f(x)dx.$$

Similarly we may define infinite repeated integrals; for instance, when $f(x, y)$ is bounded and integrable for $a + \epsilon \leqq x \leqq b$, $c + \epsilon' \leqq y \leqq d$, where ϵ and ϵ' are as small, b and d as large as we please, we have the definitions

$$\int_a^\infty dx \int_c^\infty f(x, y)dy = \lim_{\substack{\epsilon \to 0 \\ b \to \infty}} \int_{a+\epsilon}^b dx \left[\lim_{\substack{\epsilon' \to 0 \\ d \to \infty}} \int_{c+\epsilon'}^d f(x, y)dy \right],$$

$$\int_c^\infty dy \int_a^\infty f(x, y)dx = \lim_{\substack{\epsilon' \to 0 \\ d \to \infty}} \int_{e+\epsilon'}^d dy \left[\lim_{\substack{\epsilon \to 0 \\ b \to \infty}} \int_{a+\epsilon}^b f(x, y)dx \right].$$

In the following, we shall consider only integrals of the type $\int_a^\infty f(x)dx$ where $f(x)$ may cease to be bounded at $x = a$, and the corresponding case for repeated integrals. The theorems we shall prove hold, however, in the other cases; the necessary modifications of the proofs are obvious.

A convenient means of establishing the existence of an infinite integral consists in comparing it to another with positive integrand.

THEOREM XVI. *Let $f(x)$ be bounded and integrable when $a + \epsilon \leqq x \leqq b$, where ϵ is as small and b as large as we please, and let $|f(x)| \leqq M(x)$ in this interval. If $\int_{a+\epsilon}^b M(x)dx$ exists, and is bounded when $\epsilon \to 0$ and $b \to \infty$, then $\int_a^\infty f(x)dx$ exists, and*

$$\left| \int_a^\infty f(x)dx \right| \leqq \int_a^\infty M(x)dx.$$

Since $M(x) \geqq 0$, $\int_{a+\epsilon}^b M(x)dx$ increases (or remains constant) when ϵ decreases or b increases, and being bounded, this integral must therefore approach a definite limit as $\epsilon \to 0$ and $b \to \infty$, that is, $\int_a^\infty M(x)dx$ exists.

Consequently

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} \int_{a+\epsilon}^{a+\epsilon'} M(x)dx = 0, \qquad \lim_{\substack{b \to \infty \\ b' \to \infty}} \int_{b}^{b'} M(x)dx = 0,$$

and by (14) and (12)

$$\left| \int_{a+\epsilon}^{a+\epsilon'} f(x)dx \right| \leqq \int_{a+\epsilon}^{a+\epsilon'} M(x)dx, \qquad \left| \int_{b}^{b'} f(x)dx \right| \leqq \int_{b}^{b'} M(x)dx,$$

so that

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} \int_{a+\epsilon}^{a+\epsilon'} f(x)dx = 0, \qquad \lim_{\substack{b \to \infty \\ b' \to \infty}} \int_{b}^{b'} f(x)dx = 0,$$

which are the conditions for the existence of $\int_a^\infty f(x)dx$.

COROLLARY. *When* $|f(x)| < K(x-a)^{-\alpha}$, *where* $\alpha < 1$, *for* $a < x < x_1$, *and* $|f(x)| < Kx^{-\beta}$, *where* $\beta > 1$, *for* $x > x_2$, *then* $\int_a^\infty f(x)dx$ *exists.*

In fact, when ϵ and ϵ' are so small that $a + \epsilon < a + \epsilon' < x_1$, we have

$$\left| \int_{a+\epsilon}^{a+\epsilon'} f(x)dx \right| \leqq \int_{a+\epsilon}^{a+\epsilon'} K(x-a)^{-\alpha}dx = \frac{K}{1-\alpha}(\epsilon'^{1-\alpha} - \epsilon^{1-\alpha}) \to 0,$$

and for $b' > b > x_2$,

$$\left| \int_{b}^{b'} f(x)dx \right| \leqq \int_{b}^{b'} Kx^{-\beta}dx = \frac{K}{\beta-1}\left(\frac{1}{b^{\beta-1}} - \frac{1}{b'^{\beta-1}}\right) \to 0.$$

When $\int_a^\infty f(x)dx$ exists, then, by the definition of an infinite integral, $\int_a^{x_1} f(x)dx$ and $\int_{x_{n-1}}^\infty f(x)dx$ also exist, and (15) becomes

$$(29) \qquad \int_a^\infty f(x)dx = \int_a^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-2}}^{x_{n-1}} f(x)dx + \int_{x_{n-1}}^\infty f(x)dx$$

for $a < x_1 < x_2 < \cdots < x_{n-1}$. The equation of definition (16) becomes

$$(30) \qquad \int_\infty^a f(x)dx = -\int_a^\infty f(x)dx,$$

and (17) and (18) also extend immediately to the present case:

$$(31) \qquad \int_a^\infty kf(x)dx = k\int_a^\infty f(x)dx,$$

$$(32) \qquad \begin{aligned} &\int_a^\infty (f_1(x) + f_2(x) + \cdots + f_n(x))dx \\ &\qquad = \int_a^\infty f_1(x)dx + \int_a^\infty f_2(x)dx + \cdots + \int_a^\infty f_n(x)dx. \end{aligned}$$

By a passage to the limit in theorem VII, we find

Theorem XVII. *When $f(x)$ satisfies the conditions of theorem III in $(a + \epsilon, b)$ where ϵ is as small and b as large as we please, and there exists a function $F_1(x)$ continuous in $(a + \epsilon, b)$ and the derivative of which equals $f(x)$ at every point where $f(x)$ is continuous, when finally*

$$\lim_{\epsilon \to 0} F_1(a + \epsilon) = F_1(a)$$

and $\lim_{b \to \infty} F_1(b) = F_1(\infty)$ *both exist, then* $\int_a^\infty f(x)dx$ *exists and*

$$F_1(\infty) - F_1(a) = \int_a^\infty f(x)dx.$$

Similarly, it is seen that when the conditions of theorem IX are fulfilled in $(a + \epsilon, b)$, where ϵ is as small and b as large as we please, and three of the four expressions $\int_a^\infty u(x)v'(x)dx$, $\int_a^\infty v(x)u'(x)dx$, $u(a)v(a) = \lim_{\epsilon \to 0} u(a + \epsilon)v(s + \epsilon)$ and $u(\infty)v(\infty) = \lim_{b \to \infty} u(b)v(b)$ exist, then the fourth exists and

$$(33) \qquad \int_a^\infty u(x)v'(x)dx = u(\infty)v(\infty) - u(a)v(a) - \int_a^\infty v(x)u'(x)dx.$$

Finally, it is readily seen how theorem X, on integration by substitution, may be extended to the present case.

A convergent infinite series $u_0 + u_1 + \cdots + u_n + \cdots$ may be written as an infinite integral by making $f(x) = u_n$ for $n \leqq x < n + 1$. In the interval $(0, b)$, where $n + 1 \leqq b < n + 2$, $f(x)$ has the only discontinuities $x = 1, 2, \cdots, n + 1$ and is integrable by theorem III, and (15) gives

$$\int_0^b f(x)dx = \sum_{\nu=0}^{n} \int_\nu^{\nu+1} f(x)dx + \int_{n+1}^b f(x)dx = \sum_{\nu=0}^{n} u_\nu + (b - n - 1)u_{n+1}.$$

The series being convergent, $u_{n+1} \to 0$ as $n \to \infty$, so that

$$|(b - n - 1)u_{n+1}| < |u_{n+1}| \to 0.$$

In the following, the concept of *uniform approach to a limit* (or *uniform convergence*) is of fundamental importance. Let $f(x, \delta, \omega)$ depend on the parameters δ and ω; then $f(x, \delta, \omega)$ tends toward the limit $f(x)$ *uniformly* in (a, b) as $\delta \to 0$ and $\omega \to \infty$ when, for any $\epsilon > 0$ as small as we please, there exist a $\delta_0 = \delta_0(\epsilon) > 0$ and an $\omega_0 = \omega_0(\epsilon) > 0$ independent of x such that for every x in (a, b)

$$|f(x, \delta, \omega) - f(x)| < \epsilon \text{ for } 0 < \delta < \delta_0,\ \omega > \omega_0.$$

In particular, let $f(x, \delta, \omega)$ be independent of δ and make

$$f(x, \omega) = \sum_{\nu=0}^{n} u_\nu(x)$$

for $n \leqq \omega < n + 1$; we then obtain the familiar definition of uniform convergence of a series.

THEOREM XVIII. *When for* $a \leqq x \leqq b$, $f(x, \delta, \omega)$ *is bounded and integrable in respect to* x *and tends uniformly toward* $f(x)$ *as* $\delta \to 0$ *and* $\omega \to \infty$, *then* $\int_a^b f(x)dx$ *exists and*

$$\lim_{\substack{\delta \to 0 \\ \omega \to \infty}} \int_a^b f(x, \delta, \omega)dx = \int_a^b f(x)dx.$$

For any $\epsilon > 0$, there exist, by hypothesis, a $\delta_0 = \delta_0(\epsilon)$ and an $\omega_0 = \omega_0(\epsilon)$ such that $f(x, \delta, \omega) - \epsilon \leqq f(x) \leqq f(x, \delta, \omega) + \epsilon$ in (a, b) for $0 < \delta < \delta_0$, $\omega > \omega_0$. Hence $f(x)$ is bounded in (a, b) so that its upper and lower integrals exist; the application of (11) and (9) to the above inequality gives

$$\overline{\int_a^b} f(x)dx \leqq \overline{\int_a^b} f(x, \delta, \omega)dx + \epsilon(b - a),$$

$$\underline{\int_a^b} f(x)dx \geqq \underline{\int_a^b} f(x, \delta, \omega)dx - \epsilon(b - a).$$

Since $f(x, \delta, \omega)$ is integrable, its upper and lower integrals are equal, whence $0 \leqq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leqq 2\epsilon(b - a)$; but $f(x)$ being independent of ϵ, it follows that $\int_a^b f(x)dx$ exists, and the preceding inequalities become

$$-\epsilon(b - a) \leqq \int_a^b f(x)dx - \int_a^b f(x, \delta, \omega)dx \leqq \epsilon(b - a)$$

for $0 < \delta < \delta_0$, $\omega > \omega_0$, which proves our proposition.

The integral $\int_a^\infty f(x, y)dx$ is said to *exist uniformly* for y in (c, d) when $f(x, y)$ is bounded and integrable in respect to x in $(a + \epsilon, b)$ and the integral $\int_{a+\epsilon}^b f(x, y)dx$ tends toward $\int_a^\infty f(x, y)dx$ uniformly for y in (c, d), that is, when

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon' \to 0}} \int_{a+\epsilon}^{a+\epsilon'} f(x, y)dx = 0, \qquad \lim_{\substack{b \to \infty \\ b' \to \infty}} \int_b^{b'} f(x, y)dx = 0$$

uniformly for y in (c, d). A convenient test for uniform existence is contained in

THEOREM XIX. *When* $f(x, y)$ *is bounded and integrable in respect to* x *in* $(a + \epsilon, b)$, *while* y *is in* (c, d), *and when, for* $x > a$ *and* y *in* (c, d),

$|f(x, y)| < M(x)$ *where* $\int_a^\infty M(x)dx$ *exists, then* $\int_a^\infty f(x, y)dx$ *exists uniformly for* y *in* (c, d).

In fact, the proof of theorem XVI then shows that both the integrals

$$\int_{a+\epsilon}^{a+\epsilon'} f(x, y)dx \quad \text{and} \quad \int_b^{b'} f(x, y)dx$$

approach zero uniformly as $\epsilon, \epsilon' \to 0$ and $b, b' \to \infty$.

We furthermore have

THEOREM XX. *When* $f(x, y)$ *is bounded and doubly integrable for* $a + \epsilon \leqq x \leqq b$, $c \leqq y \leqq d$, *and* $\int_a^\infty f(x, y)dx$ *exists uniformly for* y *in* (c, d), *then*

$$\int_c^d dy \int_a^\infty f(x, y)dx = \int_a^\infty dx \int_c^d f(x, y)dy.$$

By theorem XIII, we have

$$\int_c^d dy \int_{a+\epsilon}^b f(x, y)dx = \int_{a+\epsilon}^b dx \int_c^d f(x, y)dy,$$

and since $\int_a^\infty f(x, y)dx$ exists uniformly for $c \leqq y \leqq d$, theorem XVIII shows that $\int_c^d dy \int_a^\infty f(x, y)dx$ exists and equals $\lim\limits_{\substack{\epsilon \to 0 \\ b \to \infty}} \int_c^d dy \int_{a+\epsilon}^b f(x, y)dx$. The preceding equation then shows that $\lim\limits_{\substack{\epsilon \to 0 \\ b \to \infty}} \int_{a+\epsilon}^b dx \int_c^d f(x, y)dy$ exists and equals $\int_c^d dy \int_a^\infty f(x, y)dx$. But the last limit, by the definition of an infinite integral, equals $\int_a^\infty dx \int_c^d f(x, y)dy$, whence our theorem.

Making $f(x, y) = u_n(y)$ for $n \leqq x < n + 1$, where all $u_n(y)$ are bounded and integrable for $c \leqq y \leqq d$, the uniform existence of $\int_0^\infty f(x, y)dy$ is equivalent to the uniform convergence of $\sum\limits_0^\infty u_n(y)$ in (c, d), whence writing x, a, b for y, c, d, we find the

COROLLARY. *When* $\sum\limits_0^\infty u_n(x)$ *converges uniformly in* (a, b), *every* $u_n(x)$ *being bounded and integrable, then*

$$\int_a^b \sum_0^\infty u_n(x)dx = \sum_0^\infty \int_a^b u_n(x)dx.$$

THEOREM XXI. *When* $f(x, y)$ *and* $\partial f(x, y)/\partial y$ *are continuous for* $a + \epsilon \leqq x \leqq b$, $c \leqq y \leqq d$, $\int_a^\infty f(x, c)dx$ *exists and* $\int_a^\infty (\partial f(x, y)/\partial y)dx$ *exists uniformly for* y *in* (c, d), *then* $\int_a^\infty f(x, y)dx$ *exists uniformly for* y *in* (c, d), *and*

$$\frac{\partial}{\partial y}\int_a^\infty f(x, y)dx = \int_a^\infty \frac{\partial f(x, y)}{\partial y}dx.$$

All conditions of theorem XIV being fulfilled for x in $(a + \epsilon, b)$ and y in (c, d), we have

$$\int_{a+\epsilon}^b f(x, y)dx = \int_{a+\epsilon}^b f(x, c)dx + \int_c^y dt \int_{a+\epsilon}^b \frac{\partial f(x, t)}{\partial t}dx.$$

Since the first integral to the right is independent of y and tends toward $\int_a^\infty f(x, c)dx$ as $\epsilon \to 0$, $b \to \infty$, and since $\int_c^y dt \int_a^\infty (\partial f(x, t)/\partial t)dx$ exists uniformly for y in (c, d) by the hypothesis on $\int_a^\infty (\partial f(x, y)/\partial y)dx$ and theorem XVIII, it follows that $\int_a^\infty f(x, y)dx$ exists uniformly for y in (c, d) and

$$\int_a^\infty f(x, y)dx = \int_a^\infty f(x, c)dx + \int_c^y dt \int_a^\infty \frac{\partial f(x, t)}{\partial t}dx.$$

From the continuity of $\partial f(x, y)/\partial y$ for $a + \epsilon \leqq x \leqq b$, $c \leqq y \leqq d$, and the uniform existence of $\int_a^\infty (\partial f(x, y)/\partial y)dx$ for y in (c, d), it follows at once that the latter integral is a continuous function of y in (c, d). Consequently, differentiating the last equation in respect to y and applying theorem VI, we obtain our proposition.

Making $f(x, y) = u_n(y)$ for $n \leqq x < n + 1$, and replacing y, c, d by x, a, b, we have the

COROLLARY. *When* $\sum_0^\infty (du_n(x)/dx)$ *converges uniformly in* (a, b), *every* $(du_n(x)/dx)$ *being continuous, and* $\sum_0^\infty u_n(a)$ *converges, then* $\sum_0^\infty u_n(x)$ *converges uniformly in* (a, b), *and*

$$\frac{d}{dx}\sum_0^\infty u_n(x) = \sum_0^\infty \frac{du_n(x)}{dx}.$$

Regarding infinite repeated integrals, we shall now prove

THEOREM XXII. *When* $f(x, y)$ *is bounded and integrable for* $a + \epsilon \leqq x \leqq b$, $c + \epsilon' \leqq y \leqq d$, *where* ϵ, ϵ' *are as small and* b, d *as large as we please,* *and* ONE *of the infinite repeated integrals* $\int_a^\infty dx \int_c^\infty |f(x, y)|\, dy$ *and* $\int_c^\infty dy \int_a^\infty |f(x, y)|\, dx$ *exists, then both the infinite repeated integrals of* $f(x, y)$ *exist and are equal:*

$$\int_a^\infty dx \int_c^\infty f(x, y)dy = \int_c^\infty dy \int_a^\infty f(x, y)dx.$$

From the existence of $\int_{a+\epsilon}^b \int_{c+\epsilon'}^d f(x, y)dxdy$ we infer, by theorem XV, that of $\int_{a+\epsilon}^b \int_{e+\epsilon'}^d |f(x, y)|\, dxdy$ and hence by theorem XIII that

$$\int_{a+\epsilon}^b dx \int_{c+\epsilon'}^d |f(x, y)|\, dy = \int_{c+\epsilon'}^d dy \int_{a+\epsilon}^b |f(x, y)|\, dx.$$

When in either of these repeated integrals ϵ and ϵ' are decreased, b and d increased, the integral cannot decrease since $|f(x, y)|$ is never negative. Consequently assuming the existence of $\int_a^\infty dx \int_c^\infty |f(x, y)|\, dy$, we have $\int_{a+\epsilon}^b dx \int_{c+\epsilon'}^d |f(x, y)|\, dy \leqq \int_a^\infty dx \int_c^\infty |f(x, y)|\, dy$, whence

$$\int_{c+\epsilon'}^d dy \int_{a+\epsilon}^b |f(x, y)|\, dx \leqq \int_a^\infty dx \int_c^\infty |f(x, y)|\, dy,$$

so that the integral to the left is bounded for all values of ϵ, ϵ', b and d considered. Since this integral does not decrease as ϵ, ϵ' decrease to zero and b, d increase indefinitely, it therefore approaches a limit not exceeding the right member, that is, $\int_c^\infty dy \int_a^\infty |f(x, y)|\, dx$ exists and

$$\int_c^\infty dy \int_a^\infty |f(x, y)|\, dx \leqq \int_a^\infty dx \int_c^\infty |f(x, y)|\, dy.$$

Having established the existence of $\int_c^\infty dy \int_a^\infty |f(x, y)|\, dx$, we apply to it the same argument as was applied to the other infinite repeated integral, and thereby establish the inequality

$$\int_a^\infty dx \int_c^\infty |f(x, y)|\, dy \leqq \int_c^\infty dy \int_a^\infty |f(x, y)|\, dx.$$

Combining the two inequalities, we find

$$\int_a^\infty dx \int_c^\infty |f(x, y)|\, dy = \int_c^\infty dy \int_a^\infty |f(x, y)|\, dx.$$

Furthermore, the integral $\int_{a+\epsilon}^{b}\int_{c+\epsilon'}^{d}(|f(x, y)| - f(x, y))dxdy$ exists, being the difference of the double integrals of $|f(x, y)|$ and $f(x, y)$, and from $0 \leqq |f(x, y)| - f(x, y) \leqq 2|f(x, y)|$ we see that

$$0 \leqq \int_{a+\epsilon}^{b} dx \int_{c+\epsilon'}^{d}(|f(x, y)| - f(x, y))dy \leqq 2\int_{a}^{\infty} dx \int_{c}^{\infty} |f(x, y)|\, dy.$$

Since the integral to the left cannot decrease as ϵ, ϵ' decrease to zero and b, d increase indefinitely, it follows that $\int_{a}^{\infty} dx \int_{c}^{\infty}(|f(x, y)| - f(x, y))dy$ exists; we may now show exactly as before that the other infinite repeated integral exists and that

$$\int_{a}^{\infty} dx \int_{c}^{\infty}(|f(x, y)| - f(x, y))dy = \int_{c}^{\infty} dy \int_{a}^{\infty}(|f(x, y)| - f(x, y))dx.$$

Our theorem now follows by subtracting this equation from that relative to $|f(x, y)|$.

Replacing $f(x, y)$ by $u_n(y)$ for $n \leqq x < n + 1$ and writing x, a for y, c we have the

COROLLARY. *When every $u_n(x)$ is finite and integrable in $(a + \epsilon, b)$ and either of the expressions $\int_{a}^{\infty}\sum_{0}^{\infty}|u_n(x)|\,dx$ and $\sum_{0}^{\infty}\int_{a}^{\infty}|u_n(x)|\,dx$ exists, then both the corresponding expressions in $u_n(x)$ exist, and*

$$\int_{a}^{\infty}\sum_{0}^{\infty}u_n(x)dx = \sum_{0}^{\infty}\int_{a}^{\infty}u_n(x)dx.$$

6. **Integrals of complex functions of a real variable.** In the following two chapters, we shall make extensive use of such integrals, which are defined in the following manner: let $f(x) = g_1(x) + ig_2(x)$, where $g_1(x)$ and $g_2(x)$ are real functions of x, bounded and integrable for $a \leqq x \leqq b$. Then, by definition,

$$\int_{a}^{b} f(x)dx = \int_{a}^{b} g_1(x)dx + i\int_{a}^{b} g_2(x)dx,$$

and the double and repeated integrals of $f(x, y) = g_1(x, y) + ig_2(x, y)$, as well as the various kinds of infinite integrals, are defined in the same manner by decomposition into their real and imaginary parts.

Theorems II and XI, as well as formulas (7)–(13), apply to real functions only; but all the remaining theorems and formulas may be extended to the present case. In fact, the proof of theorem I subsists unchanged, since it makes no use of the reality of $f(x_1, \cdots, x_n)$. Theorem III is immediately extended by decomposition of $f(x)$ into its real and complex parts. In theorem IV, let M_ν, m_ν, $M_{1\nu}$, $m_{1\nu}$, $M_{2\nu}$, $m_{2\nu}$ be the upper and lower bounds of $|f(x)|$, $|g_1(x)|$ and $|g_2(x)|$ in $(x_{\nu-1}, x_\nu)$, and x', x'' any two points in this interval. We have

$$\begin{aligned}|f(x')| - |f(x'')| \leqq |f(x') - f(x'')| &\\ &= |g_1(x') - g_1(x'') + i(g_2(x') - g_2(x''))| \\ &\leqq |g_1(x') - g_1(x'')| + |g_2(x') - g_2(x'')| \\ &\leqq (M_{1\nu} - m_{1\nu}) + (M_{2\nu} - m_{2\nu}),\end{aligned}$$

and since for suitably chosen values of x' and x'' $|f(x')|$ will differ from M_ν, and $|f(x'')|$ from m_ν, as little as we please, it follows that

$$M_\nu - m_\nu \leqq (M_{1\nu} - m_{1\nu}) + (M_{2\nu} - m_{2\nu}),$$

and consequently the $S - s$ relative to $|f(x)|$ may be made as small as we please, since it is less than the sum of the $S - s$ relative to $|g_1(x)|$ and that relative to $|g_2(x)|$, both of which may be made as small as we please by theorem IV. Theorems V–X, XII–XIV are immediately extended by decomposition (it being observed that in X $\varphi(t)$ is a real function of the real variable t), XV is extended in the same way as IV, and the proof of XVI remains unchanged, since (14) subsists in the complex case. In fact, with the notations of theorem VIII, we have, since $x_\nu - x_{\nu-1} > 0$, for $f(x)$ complex

$$|\Sigma f(\xi_\nu)(x_\nu - x_{\nu-1})| \leqq \Sigma\, |f(\xi_\nu)|\, (x_\nu - x_{\nu-1});$$

as we have seen that theorem VIII extends to the complex case, it follows that for $\delta \to 0$, the left side approaches $\left|\int_a^b f(x)dx\right|$, and the right, $\int_a^b |f(x)|\, dx$, whence we obtain (14). In exactly the same way, we extend (26), while (15), (16), (17), where k may now be a complex constant, (18), (27), (28), (32), (33) are extended immediately by decomposition. Theorems XVII and XVIII extend by decomposition, the proof of XIX remains unchanged, XX and XXI (with corollaries) are again extended by decomposition, while in XXII (and corollary) we first conclude from $|g_1(x, y)| \leqq |f(x, y)|$ and $|g_2(x, y)| \leqq |f(x, y)|$ that the same infinite repeated integrals of $|g_1(x, y)|$ and $|g_2(x, y)|$ exist as that of $|f(x, y)|$ included in the hypothesis; the extension is then finished by decomposition.

We shall finally prove the following

THEOREM XXIII. *When, for an $\epsilon > 0$ as small and a b as large as we please, the function $f(s, t)$ of two complex variables s and t is holomorphic for*

$a+\epsilon \leqq t \leqq b$ and $|s-s_0| \leqq r$, where r is independent of ϵ and b, and when finally $\int_a^\infty f(s,t)dt$ converges uniformly for $|s-s_0|<r$, then this integral defines a function of s which is holomorphic for $|s-s_0|<r$.

Write $s-s_0=x+iy$ and $f(s,t)=g_1(x,y,t)+ig_2(x,y,t)$, and denote by P the pointset defined by $a+\epsilon \leqq t \leqq b$, $x^2+y^2 \leqq r^2$; since $f(s,t)$ is holomorphic on P, g_1 and g_2 are continuous on P and this pointset being closed, g_1 and g_2 are uniformly continuous on P (theorem I). Since g_1 and g_2 are continuous, we may take the whole interval $(a+\epsilon, b)$ as Δ'' in theorem III, and on account of the uniform continuity on P, we may choose, for any $\epsilon'>0$, a δ *independent* of x and y such that S and s differ by less than ϵ' when all subdivisions of $(a+\epsilon, b)$ are less than δ. Consequently, as $\delta \to 0$, the sum

$$\Sigma f(s,\tau_\nu)(t_\nu-t_{\nu-1}) = \Sigma g_1(x,y,\tau_\nu)(t_\nu-t_{\nu-1}) + i\Sigma g_2(x,y,\tau_\nu)(t_\nu-t_{\nu-1}),$$

formed according to theorem VIII, tends toward $\int_{a+\epsilon}^b f(s,t)dt$ uniformly for $|s-s_0| \leqq r$. But each $f(s,\tau_\nu)$ is expansible in a power series in $s-s_0$ convergent for $|s-s_0| \leqq r$, and therefore also $\Sigma f(s,\tau_\nu)(t_\nu-t_{\nu-1})$ which contains a finite number of such terms. Let $\delta_1, \delta_2, \cdots$ be a sequence of positive numbers such that $\delta_n \to 0$ as $n \to \infty$, and let

$$\Phi_n(s;\epsilon,b) = \Sigma f(s,\tau_\nu)(t_\nu-t_{\nu-1})$$

for a certain mode of subdivision where all $t_\nu - t_{\nu-1} < \delta_n$; since $\Phi_n(s;\epsilon,b)$ is a power series in $s-s_0$ convergent for $|s-s_0| \leqq r$ and since $\Phi_n(s;\epsilon,b)$ tends uniformly toward $\int_{a+\epsilon}^b f(s,t)dt$, we conclude from Weierstrass' theorem on sequences of analytic functions that $\int_{a+\epsilon}^b f(s,t)dt$ is expansible in a power series in $s-s_0$ convergent for $|s-s_0| \leqq r$. Now choose sequences $\epsilon_1, \epsilon_2, \cdots$ and $b_1, b_2, \cdots$ such that $\epsilon_n \to 0$ and $b_n \to \infty$ as $n \to \infty$; since $\int_a^\infty f(s,t)dt$ converges uniformly for $|s-s_0| \leqq r$, the expressions $\int_{a+\epsilon_n}^{b_n} f(s,t)dt$, which are power series in $s-s_0$ convergent for $|s-s_0| \leqq r$, converge uniformly toward the limit $\int_a^\infty f(s,t)dt$ which is therefore, by a second application of Weierstrass' theorem, a power series convergent for $|s-s_0|<r$.

7. **Fourier series.** Let $f(x)$ be bounded and integrable for $\epsilon \leqq x \leqq 1-\epsilon$, where ϵ is as small as we please, and suppose that $\int_0^1 |f(x)|\,dx$ exists.

Since $|f(x) \sin 2n\pi x| \leqq |f(x)|$ and $|f(x) \cos 2n\pi x| \leqq |f(x)|$, it follows from theorem XVI that the integrals

$$a_n = 2\int_0^1 f(x) \cos 2n\pi x dx, \qquad b_n = 2\int_0^1 f(x) \sin 2n\pi x dx \tag{34}$$

both exist. Then the Fourier series of $f(x)$ is defined as

$$\frac{a_0}{2} + \sum_1^\infty (a_n \cos 2n\pi x + b_n \sin 2n\pi x). \tag{35}$$

In regard to the convergence of this series, we shall need in the following only this very special theorem:

When $f(x)$ is holomorphic for $0 < x < 1$ (but not necessarily at the end points of the interval) and $\int_0^1 |f(x)|\, dx$ exists, the Fourier series of $f(x)$ converges to the sum $f(x)$ uniformly for $\epsilon \leqq x \leqq 1 - \epsilon$, where $\epsilon > 0$ but as small as we please.

Using the integration variable t instead of x in (34), we find

$$\begin{aligned} a_n \cos 2n\pi x + b_n \sin 2n\pi x \\ &= 2\int_0^1 f(t)(\cos 2n\pi t \cos 2n\pi x + \sin 2n\pi t \sin 2n\pi x)dt \\ &= 2\int_0^1 f(t) \cos 2n\pi(t - x)dt, \end{aligned}$$

and since we have[3]

$$1 + 2\sum_{\nu=1}^n \cos 2\nu\pi(t - x) = \frac{\sin (2n + 1)\pi(t - x)}{\sin \pi(t - x)}$$

it follows that

$$\begin{aligned} \frac{a_0}{2} + \sum_{\nu=1}^n (a_\nu \cos 2\nu\pi x + b_\nu \sin 2\nu\pi x) \\ &= \int_0^1 f(t) \left[1 + 2\sum_{\nu=1}^n \cos 2\nu\pi(t - x) \right] dt \\ &= \int_0^1 f(t) \frac{\sin (2n + 1)\pi(t - x)}{\sin \pi(t - x)} dt. \end{aligned} \tag{36}$$

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[3] Proof: in the finite geometric series

$$\sum_{\nu=0}^n e^{(\alpha+2\nu\beta)i} = e^{\alpha i} \cdot \frac{e^{2(n+1)\beta i} - 1}{e^{2\beta i} - 1} = e^{\alpha i} \frac{e^{(2n+1)\beta i} - e^{-\beta i}}{e^{\beta i} - e^{-\beta i}}$$

$$= \frac{e^{(\alpha+(2n+1)\beta)i} - e^{(\alpha-\beta)i}}{2i \sin \beta}$$

compare the real and imaginary parts on both sides:

$$\sum_{\nu=0}^n \cos (\alpha + 2\nu\beta) = \frac{\sin (\alpha + (2n + 1)\beta) - \sin (\alpha - \beta)}{2 \sin \beta},$$

$$\sum_{\nu=0}^n \sin (\alpha + 2\nu\beta) = \frac{\cos (\alpha + \beta) - \cos (\alpha + (2n + 1)\beta)}{2 \sin \beta}.$$

From the first of these, we obtain the formula in the text by making $\alpha = 0$, $\beta = \pi(t - x)$.

In the special case $f(x) = 1$, (34) gives $a_0 = 2$, $a_\nu = b_\nu = 0$ $(\nu > 0)$, and (36) becomes

$$1 = \int_0^1 \frac{\sin (2n + 1)\pi(t - x)}{\sin \pi(t - x)} dt;$$

multiplying this by $f(x)$ and subtracting from (36), it is seen that

$$\begin{aligned} &\frac{a_0}{2} + \sum_{\nu=1}^{n} (a_\nu \cos 2\nu\pi x + b_\nu \sin 2\nu\pi x) - f(x) \\ (37) \qquad &= \int_0^1 (f(t) - f(x)) \frac{\sin (2n + 1)\pi(t - x)}{\sin \pi(t - x)} dt \\ &= J_1 + J_2 + J_3, \end{aligned}$$

where J_1, J_2 and J_3 are the integrals corresponding to the decomposition of the interval $(0, 1)$ into the three intervals $(0, \delta)$, $(\delta, 1 - \delta)$ and $(1 - \delta, 1)$. Our theorem will then be proved if we show that after assigning an arbitrarily small ϵ, a δ may be chosen sufficiently small and an n_0 sufficiently large, both independent of x, such that $| J_1 | + | J_2 | + | J_3 |$ will be as small as we please for $n > n_0$ and every x in $(\epsilon, 1 - \epsilon)$. Assume $0 < \delta < \epsilon/2$; for $\epsilon \leqq x \leqq 1 - \epsilon$ and $0 \leqq t \leqq \delta$ or $1 - \delta \leqq t \leqq 1$, we then have $\epsilon/2 < | t - x | < 1 - (\epsilon/2)$ and consequently $| \sin \pi(t - x) | > \sin (\pi\epsilon/2)$; since $|f(t) - f(x) | \leqq | f(t) | + | f(x) |$, the expression under the integral sign in J_1 and J_3 will be less than $(| f(t) | + | f(x) |)/\sin \pi(\epsilon/2)$, whence

$$| J_1 | + | J_3 | < \frac{1}{\sin (\pi\epsilon/2)} \left[\int_0^\delta (| f(t) | + | f(x) |) dt + \int_{1-\delta}^1 (| f(t) | + | f(x) |) dt \right].$$

As $f(x)$ is holomorphic in $(\epsilon, 1 - \epsilon)$, there exists a constant M such that $| f(x) | < M$ for all x in $(\epsilon, 1 - \epsilon)$, whence

$$| J_1 | + | J_3 | < \frac{1}{\sin (\pi\epsilon/2)} \left(2\delta M + \int_0^\delta | f(t) | \, dt + \int_{1-\delta}^1 | f(t) | \, dt \right),$$

Since $\int_0^1 | f(t) | \, dt$ exists by hypothesis, the last two integrals may be made as small as we please by taking δ sufficiently small, and the same being true of $2\delta M$, it follows that $| J_1 | + | J_3 |$ may be made as small as we please by taking δ sufficiently small and independent of x. In J_2, the function

$$\frac{f(t) - f(x)}{\sin \pi(t - x)} = \varphi(x, t)$$

is evidently holomorphic for $\epsilon \leqq x \leqq 1 - \epsilon$, $\delta \leqq t \leqq 1 - \delta$, so that constants M_1 and M_2 exist such that $|\varphi(x, t)| < M_1$ and

$$\left|\frac{\partial\varphi(x, t)}{\partial t}\right| < M_2$$

for all these values of x and t. Integrating by parts,

$$\begin{aligned} J_2 = \int_\delta^{1-\delta} & \varphi(x, t) \sin (2n + 1)\pi(t - x)dt \\ & = -\left[\frac{1}{(2n+1)\pi}\varphi(x, t) \cos (2n + 1)\pi(t - x)\right]_{t=\delta}^{t=1-\delta} \\ & + \frac{1}{(2n+1)\pi}\int_\delta^{1-\delta}\frac{\partial\varphi(x, t)}{\partial t}\cos (2n + 1)\pi(t - x)dt, \end{aligned}$$

and since $|\varphi(x, t) \cos (2n + 1)\pi(t - x)| \leqq |\varphi(x, t)| < M_1$,

$$\left|\frac{\partial\varphi(x, t)}{\partial t}\cos (2n + 1)\pi(t - x)\right| < M_2,$$

we have

$$|J_2| < \frac{2M_1 + M_2}{(2n+1)\pi}.$$

Consequently, after fixing a δ so small that $|J_1| + |J_2| < \eta$, where η is as small as we please, M_1 and M_2, which depend on δ but not on x, are fixed, and the last inequality now permits us to determine an n_0 (dependent on δ but independent of x) such that $|J_2| < \eta$ for $n \geqq n_0$ and all x in $(\epsilon, 1 - \epsilon)$. The expression to the left in (37) is then less than 2η in absolute value, which proves our theorem.

We shall now derive a formula which is useful in estimating the remainder in some series occurring in the next chapters: Let

$$c_n \geqq c_{n+1} \geqq \cdots \geqq c_{n+p-1} \geqq c_{n+p} \geqq 0,$$

k a constant and $\epsilon > 0$ but as small as we please, then

$$\left|\sum_{\nu=n}^{n+p} c_\nu \sin (2\nu + k)\pi x\right| \leqq \frac{c_n}{\sin \pi\epsilon}, \qquad \epsilon \leqq x \leqq 1 - \epsilon. \tag{38}$$

Using the identity

$$2 \sin \pi x \sin (2\nu + k)\pi x = \cos (2\nu - 1 + k)\pi x - \cos (2\nu + 1 + k)\pi x,$$

we have

$$\begin{aligned} & 2 \sin \pi x \sum_{\nu=n}^{n+p} c_\nu \sin (2\nu + k)\pi x \\ & \qquad = \sum_{\nu=n}^{n+p} c_\nu \cos (2\nu - 1 + k)\pi x - \sum_{\nu=n}^{n+p} c_\nu \cos (2\nu + 1 + k)\pi x \end{aligned}$$

$$= c_n \cos (2n - 1 + k)\pi x - \sum_{\nu=n+1}^{n+p} (c_{\nu-1} - c_\nu) \cos (2\nu - 1 + k)\pi x$$
$$- c_{n+p} \cos (2n + 2p + 1 + k)\pi x,$$

and taking the sum of the absolute values of all terms to the right and remembering that $c_{\nu-1} - c_\nu \geqq 0$,

$$\left| 2 \sin \pi x \sum_{\nu=n+1}^{n+p} c_\nu \sin (2\nu + k)\pi x \right| \leqq c_n + \sum_{\nu=n+1}^{n+p} (c_{\nu-1} - c_\nu) + c_{n+p} = 2c_n,$$

whence (38) follows at once.

8. **Bernoulli functions and numbers.** We now apply the theorem of the preceding paragraph to $f(x) = \frac{1}{2} - x$; (34) gives, upon integration by parts, $a_0 = 0$, $a_n = 0$, $b_n = 1/n\pi$ $(n \geqq 1)$, whence

$$\tfrac{1}{2} - x = \sum_1^\infty \frac{\sin 2n\pi x}{n\pi}, \qquad 0 < x < 1,$$

the series being uniformly convergent for $\epsilon \leqq x \leqq 1 - \epsilon$. The series to the right converges also for $x = 0$ and $x = 1$, the sum being zero in either case, while the left side becomes $\pm \frac{1}{2}$. Let $[x]$ be the greatest integer contained in x, so that $0 \leqq x - [x] < 1$, then $\frac{1}{2} + [x] - x$ remains unchanged when an integer is added to x, and equals $\frac{1}{2} - x$ for $0 < x < 1$. Since the Fourier series to the right also has the period unity, we see that if we write

$$P_1(x) = \tfrac{1}{2} + [x] - x, \tag{39}$$

then for any x which is not an integer

$$P_1(x) = \sum_1^\infty \frac{\sin 2n\pi x}{n\pi}, \tag{40}$$

while for an integer m, $P_1(m + \delta) \to \frac{1}{2}$ and $P_1(m - \delta) \to -\frac{1}{2}$ as $\delta \to 0$ through positive values. Now write

$$P_2(x) = \sum_1^\infty \frac{\cos 2n\pi x}{2n^2\pi^2},$$

the series being uniformly convergent for all values of x, since its general term is less than $1/n^2$ in absolute value. Then, since (40) converges uniformly for $\epsilon \leqq x \leqq 1 - \epsilon$, we have

$$\frac{dP_2(x)}{dx} = - P_1(x) = x - \tfrac{1}{2}$$

(theorem XXI cor.) and therefore

$$P_2(x) = \frac{x^2 - x}{2} + c$$

in the latter interval; to determine the constant of integration, we observe that, on account of the uniform convergence of the series,

$$\int_0^1 P_2(x)dx = \int_0^1 \sum_1^\infty \frac{\cos 2n\pi x}{2n^2\pi^2}dx = \sum_1^\infty \int_0^1 \frac{\cos 2n\pi x}{2n^2\pi^2}dx = 0,$$

and therefore

$$\int_0^1 \left(\frac{x^2 - x}{2} + c\right)dx = 0 \qquad \text{or} \qquad c = \tfrac{1}{12}.$$

Now the polynomial $\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$ takes the same value $(= \frac{1}{12})$ for $x = 0$ and $x = 1$; consequently the expression $\frac{1}{2}(x - [x])^2 - \frac{1}{2}(x - [x]) + \frac{1}{12}$ admits the period unity and is continuous for all values of x, so that the relation

$$P_2(x) = \tfrac{1}{2}(x - [x])^2 - \tfrac{1}{2}(x - [x]) + \tfrac{1}{12} = \sum_1^\infty \frac{\cos 2n\pi x}{2n^2\pi^2},$$

which we proved for $0 < x < 1$, subsists for *all* values of x since both members are continuous everywhere and admit the period unity. We observe that the integral values of x form no exception as in the case of $P_1(x)$.

Continuing this process, we define functions $P_k(x)$, $k = 2, 3, \cdots$, by the relations

$$P_k(x) = (-1)^k \frac{dP_{k+1}(x)}{dx},$$

$$\int_0^1 P_{k+1}(x)dx = 0, \tag{41}$$

$$P_{k+1}(x + 1) = P_{k+1}(x),$$

and show by complete induction that $P_k(x)$ is a polynomial of degree k in $x - [x]$ with rational coefficients, that $P_k(x)$ is continuous for all values of x when $k > 1$, since $P_{k+1}(1) - P_{k+1}(0) = (-1)^k \int_0^1 P_k(x)dx = 0$, and that

$$P_{2k}(x) = \sum_1^\infty \frac{\cos 2n\pi x}{2^{2k-1}n^{2k}\pi^{2k}},$$

$$P_{2k+1}(x) = \sum_1^\infty \frac{\sin 2n\pi x}{2^{2k}n^{2k+1}\pi^{2k+1}}, \tag{42}$$

these series being absolutely and uniformly convergent for all values of x.

From (42) it follows that, when m is an integer

$$
\begin{aligned}
&P_{2k+1}(m) = P_{2k+1}(m+\tfrac{1}{2}) = 0,\\
(43)\qquad &P_{2k}(m) = \frac{1}{2^{2k-1}\pi^{2k}}\sum_1^\infty \frac{1}{n^{2k}},\\
&P_{2k}(m+\tfrac{1}{2}) = -\left(1-\frac{1}{2^{2k-1}}\right)P_{2k}(m),
\end{aligned}
$$

the last relation resulting from the identity

$$\sum_1^\infty \frac{(-1)^n}{n^{2k}} = 2\sum_1^\infty \frac{1}{(2n)^{2k}} - \sum_1^\infty \frac{1}{n^{2k}} = -\left(1-\frac{1}{2^{2k-1}}\right)\sum_1^\infty \frac{1}{n^{2k}}.$$

Furthermore (42) gives

$$(44)\qquad P_k(1-x) = (-1)^k P_k(x) = P_k(-x).$$

We shall now show that *for k odd and > 1, the equation $P_k(x) = 0$ has the roots $0, \frac{1}{2}, 1$ and that $P_k(x) > 0$ for $0 < x < \frac{1}{2}$, $P_k(x) < 0$ for $\frac{1}{2} < x < 1$, while for k even, there is one and only one root interior to each of the intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$.* From (43) it is seen that $P_k(0) = P_k(\frac{1}{2}) = P_k(1) = 0$ for k odd, but that for k even none of these expressions vanishes. For $k = 2$, our theorem is true, since $P_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + 12$ in $(0, 1)$; we may therefore assume the theorem proved up to the index $k-1$ and deduce its validity for the index k in the following manner. If, for k odd, $P_k(x) = 0$ has a root in $(0, 1)$ distinct from $0, \frac{1}{2}, 1$, we may assume, by (44), that it is interior to $(0, \frac{1}{2})$. But since

$$P_{k-1}(x) = \frac{dP_k(x)}{dx}$$

by (41), $P_{k-1}(x) = 0$ would then, according to Rolle's theorem, have two roots interior to the interval $(0, \frac{1}{2})$, contrary to the proven facts for $k-1$ even. Thus $P_k(x)$ does not change its sign for $0 < x < \frac{1}{2}$, and since for $x = 0$,

$$\frac{dP_k(x)}{dx} = P_{k-1}(0)$$

is positive by (43), we find that $P_k(x) > 0$ for $0 < x < \frac{1}{2}$, and (44) then gives $P_k(x) < 0$ for $\frac{1}{2} < x < 1$. If, for k even, $P_k(x) = 0$ has two roots interior to $(0, \frac{1}{2})$, $P_{k-1}(x) = 0$ would have a root interior to the same interval, contrary to the proven facts for $k-1$ odd. The same argument applies to the interval $(\frac{1}{2}, 1)$.

The expressions $P_{2k}(m) = P_{2k}(0)$ of (43) also occur in the expansion of the cotangent in a power series. From the decomposition of the cotangent into partial fractions we obtain

$$\frac{t}{2}\cot\frac{t}{2} = 1 - \sum_{n=1}^{\infty}\frac{2t^2}{4\pi^2n^2 - t^2},$$

the series being uniformly convergent for $|t| \leqq 2\pi - \epsilon$. For such values of t, we have

$$\frac{2t^2}{4\pi^2n^2 - t^2} = \sum_{=1}^{\infty}\frac{t^{2k}}{2^{2k-1}\pi^{2k}n^{2k}},$$

and by Weierstrass's theorem on series of analytic functions, we obtain

$$\frac{t}{2}\cot\frac{t}{2} = 1 - \sum_{k=1}^{\infty}t^{2k}\cdot\sum_{n=1}^{\infty}\frac{1}{2^{2k-1}\pi^{2k}n^{2k}}.$$

We now define the kth Bernoulli number B_k by[4]

$$B_k = \frac{(2k)!}{2^{2k-1}\pi^{2k}}\sum_{n=1}^{\infty}\frac{1}{n^{2k}}, \tag{45}$$

and obtain[5]

$$\frac{t}{2}\cot\frac{t}{2} = 1 - \sum_{k=1}^{\infty}\frac{B_kt^{2k}}{(2k)!}. \tag{46}$$

In

$$\frac{t}{2}\cot\frac{t}{2} = \frac{ti}{2}\,\frac{e^{ti/2} + e^{-ti/2}}{e^{ti/2} - e^{-ti/2}} = \frac{ti}{2} + \frac{ti}{e^{ti} - 1}$$

we now replace t by $-ti$; (46) then becomes

$$1 - \frac{t}{e^t - 1} = \frac{t}{2} + \sum_{k=1}^{\infty}\frac{(-1)^kB_kt^{2k}}{(2k)!}, \qquad |t| < 2\pi. \tag{47}$$

Multiplying both sides by

$$e^t - 1 = \sum_1^{\infty}\frac{t^k}{k!},$$

we find

$$\sum_1^{\infty}\frac{t^k}{k!} - t = \left(\sum_1^{\infty}\frac{t^k}{k!}\right)\left(\frac{t}{2} + \sum_1^{\infty}\frac{(-1)^kB_kt^{2k}}{(2k)!}\right)$$

and comparing coefficients of t^{2k+1} on both sides

$$\frac{1}{(2k+1)!} = \frac{1}{2(2k)!} - \frac{B_1}{(2k-1)!2!} + \frac{B_2}{(2k-3)!4!} - \frac{B_3}{(2k-5)!6!} + \cdots + \frac{(-1)^kB_k}{1!(2k)!};$$

---

[4] These numbers appear for the first time in Jacob Bernoulli's Ars conjectandi, Basel, 1713, p. 97, in connection with the problem of expressing the sum $1^k + 2^k + 3^k + \cdots + (n-1)^k$ as a polynomial in n.

[5] Euler, L., Introductio in analysis infinitorum, Lausanne, 1748, p. 161.

this recurrent formula allows us to calculate successively $B_1, B_2, \cdots$ which are clearly all rational numbers. The first ten Bernoulli numbers are thus found to be

$$B_1 = \tfrac{1}{6},\quad B_2 = \tfrac{1}{30},\quad B_3 = \tfrac{1}{42},\quad B_4 = \tfrac{1}{30},\quad B_5 = \tfrac{5}{66},\quad B_6 = \tfrac{691}{2730},$$
$$B_7 = \tfrac{7}{6},\quad B_8 = \tfrac{3617}{510},\quad B_9 = \tfrac{43867}{798},\quad B_{10} = \tfrac{174611}{330}.$$

The comparison of (43) and (45) gives, since $P_k(m) = P_k(0)$ for $k > 1$,

$$P_{2k+1}(0) = P_{2k+1}(\tfrac{1}{2}) = 0,$$
$$P_{2k}(0) = \frac{1}{(2k)!}B_k, \qquad P_{2k}(\tfrac{1}{2}) = -\frac{1}{(2k)!}\left(1 - \frac{1}{2^{2k-1}}\right)B_k. \tag{48}$$

The left side of (47) is the special case $x = 0$ of $1 - [(te^{tx})/(e^t - 1)]$, and the latter expression may evidently be expanded in a power series in t, the coefficients being polynomials in x, the series converging uniformly for $|t| \leqq 2\pi - \epsilon$ and $|x| < a$, where ϵ is as small and a as large as we please. We write this expansion

$$1 - \frac{te^{tx}}{e^t - 1} = \sum_{k=1}^{\infty}(-1)^{k(k-1)/2}\varphi_k(x)t^k. \tag{49}$$

Since

$$1 - \frac{te^{tx}}{e^t - 1} = 1 + e^{tx}\left(1 - \frac{t}{e^t - 1} - 1\right),$$

we obtain from (47) and (49)

$$\sum_1^{\infty}(-1)^{k(k-1)/2}\varphi_k(x)t^k = 1 + \left(\sum_0^{\infty}\frac{x^kt^k}{k!}\right)\left(-1 + \frac{t}{2} + \sum_1^{\infty}(-1)^k\frac{B_kt^{2k}}{(2k)!}\right),$$

and upon multiplication and comparison of coefficients of like powers of t,

$$\begin{aligned}(-1)^k\varphi_{2k}(x) = &-\frac{x^{2k}}{(2k)!} + \frac{x^{2k-1}}{2(2k-1)!} - \frac{B_1x^{2k-2}}{(2k-2)!2!} + \frac{B_2x^{2k-4}}{(2k-4)!4!}\\ &- \frac{B_3x^{2k-6}}{(2k-6)!6!} + \cdots + (-1)^{k-1}\frac{B_{k-1}x^2}{2!(2k-2)!}\\ &+ (-1)^k\frac{B_k}{(2k)!},\end{aligned}$$

$$\begin{aligned}(-1)^k\varphi_{2k+1}(x) = &-\frac{x^{2k+1}}{(2k+1)!} + \frac{x^{2k}}{2(2k)!} - \frac{B_1x^{2k-1}}{(2k-1)!2!} + \frac{B_2x^{2k-3}}{(2k-3)!4!}\\ &- \frac{B_3x^{2k-5}}{(2k-5)!6!} + \cdots + (-1)^{k-1}\frac{B_{k-1}x^3}{3!(2k-2)!}\\ &+ (-1)^k\frac{B_kx}{(2k)!}\end{aligned}$$

for $k > 0$, while $\varphi_1(x) = \frac{1}{2} - x$, so that for $0 < x < 1$, $\varphi_1(x) = P_1(x)$. We shall now prove that for $k > 1$ and $0 \leqq x \leqq 1$

$$\varphi_k(x) = P_k(x).$$

Since for $0 \leqq x \leqq 1$, the $P_k(x)$ are completely determined by the first two relations (41), it is evidently sufficient to show that the $\varphi_k(x)$ satisfy the same relations. Differentiating (49) in respect to x, we obtain, since the differentiation of the right hand series term by term is legitimate according to Weierstrass's theorem on series of analytic functions,

$$\sum_1^\infty (-1)^{k(k-1)/2} \frac{d\varphi_k(x)}{dx} t^k = -\frac{t^2 e^{tx}}{e^t - 1} = t\left(\sum_1^\infty (-1)^{k(k-1)/2} \varphi_k(x) t^k - 1\right),$$

and comparing coefficients of t^{k+1}

$$(-1)^k \frac{d\varphi_{k+1}(x)}{dx} = \varphi_k(x).$$

Integrating (49) from $x = 0$ to $x = 1$, we find

$$\sum_1^\infty (-1)^{k(k-1)/2} t^k \int_0^1 \varphi_k(x) dx = \left[x - \frac{e^{tx}}{e^t - 1}\right]_{x=0}^{x=1} = 0,$$

whence

$$\int_0^1 \varphi_k(x) dx = 0.$$

Outside of (0, 1), $\varphi_k(x)$ and $P_k(x)$ do not coincide, since $P_k(x)$ has the period unity, but $\varphi_k(x)$ has not. To find the difference $\varphi_k(x + 1) - \varphi_k(x)$, change x into $x + 1$ in (49) and subtract:

$$\sum_1^\infty (-1)^{k(k-1)/2} (\varphi_k(x + 1) - \varphi_k(x)) t^k = -te^{tx} = -\sum_1^\infty \frac{x^{k-1} t^k}{(k - 1)!},$$

whence

$$\varphi_k(x + 1) - \varphi_k(x) = -(-1)^{k(k-1)/2} \frac{x^{k-1}}{(k - 1)!}.$$

Changing k into $k + 1$ and adding these equations for $x = 0, 1, 2, \cdots, n - 1$, we find

$$1^k + 2^k + 3^k + \cdots + (n - 1)^k = (-1)^{k(k-1)/2} k!(\varphi_{k+1}(n) - \varphi_{k+1}(0)),$$

which is essentially[6] the formula established by Bernoulli.

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[6] The Bernoulli polynomials (as distinct from the periodic Bernoulli *functions* $P_k(x)$) are commonly defined by

$$t\frac{e^{tx} - 1}{e^t - 1} = \sum_1^\infty \bar{\varphi}_k(x) \frac{t^k}{k!},$$

and by comparison to (49), we readily find the relation $\bar{\varphi}_k(x) = (-1)^{k(k-1)/2}(\varphi_k(0) - \varphi_k(x))$ between this definition and that of the text; the latter has been chosen in view of its application to Euler's summation formula in the next paragraph. For further information on Bernoulli numbers and polynomials, see Saalschütz, L., Vorlesungen über die Bernoullischen Zahlen, Berlin, 1893.

9. Euler's summation formula. Let $F(s)$ be a real or complex function of the real variable s, having continuous derivatives up to the order $2m$ inclusive, or we may suppose s complex and $F(s)$ an analytic function; furthermore let $0 \leqq a < 1$ and $b > 0$. In the identity

$$F(s + nb) - F(s + ab + \nu b) = b \int_{a+\nu}^{n} F'(s + tb)dt$$

make $\nu = 0, 1, 2, \cdots, n - 1$ and add, obtaining

$$nF(s + nb) - \sum_{\nu=0}^{n-1} F(s + ab + \nu b) = b \sum_{\nu=0}^{n-1} \int_{\nu+a}^{n} F'(s + tb)dt.$$

Decomposing each integral by the formula

$$\int_{\nu+a}^{n} F'(s + tb)dt = \int_{\nu+a}^{\nu+1} F'(s + tb)dt + \sum_{\mu=\nu+1}^{n-1} \int_{\mu}^{\mu+1} F'(s + tb)dt$$

we obtain

$$nF(s + nb) - \sum_{\nu=0}^{n-1} F(s + ab + \nu b)$$
$$= b \sum_{\nu=0}^{n-1} \int_{\nu+a}^{\nu+1} F'(s + tb)dt + b \sum_{\nu=0}^{n-1} \sum_{\mu=\nu+1}^{n-1} \int_{\mu}^{\mu+1} F'(s + tb)dt;$$

in the double sum, each integral between limits μ and $\mu + 1$ occurs μ times, namely for $\nu = 0, 1, 2, \cdots, \mu - 1$, so that

$$\sum_{\nu=0}^{n-1} \sum_{\mu=\nu+1}^{n-1} \int_{\mu}^{\mu+1} F'(s + tb)dt = \sum_{\mu=1}^{n-1} \mu \int_{\mu}^{\mu+1} F'(s + tb)dt,$$

or adding to the last sum the zero term corresponding to $\mu = 0$, and writing ν instead of μ

$$nF(s + nb) - \sum_{\nu=0}^{n-1} F(s + ab + \nu b)$$
$$= b \sum_{\nu=0}^{n-1} \left(\int_{\nu+a}^{\nu+1} F'(s + tb)dt + \nu \int_{\nu}^{\nu+1} F'(s + tb)dt \right).$$

In the integrals with lower limit $\nu + a$, we may reduce this limit to ν by observing that for $\nu \leqq t < \nu + a$, $[t - a] = \nu - 1$, or $[t - a] - \nu + 1 = 0$, while for $\nu + a \leqq t < \nu + 1$, $[t - a] = \nu$, or $[t - a] - \nu + 1 = 1$; consequently

$$\int_{\nu}^{\nu+1} ([t - a] - \nu + 1)F'(s + tb)dt = \int_{\nu+a}^{\nu+1} F'(s + tb)dt,$$

and substituting in the previous equation

$$nF(s + nb) - \sum_{\nu=0}^{n-1} F(s + ab + \nu b) = b \sum_{\nu=0}^{n-1} \int_{\nu}^{\nu+1} ([t - a] + 1)F'(s + tb)dt$$
$$= b \int_{0}^{n} ([t - a] + 1)F'(s + tb)dt.$$

But from (39), $[t - a] + 1 = t - a + \frac{1}{2} + P_1(t - a)$, and integrating by parts, we find

$$b \int_0^n (t - a + \tfrac{1}{2})F'(s + tb)dt$$
$$= (n - a + \tfrac{1}{2})F(s + nb) - (\tfrac{1}{2} - a)F(s) - \int_0^n F(s + tb)dt;$$

introducing all this in the preceding formula, we obtain

$$\sum_{\nu=0}^{n-1} F(s + ab + \nu b) = \int_0^n F(s + tb)dt + (\tfrac{1}{2} - a)(F(s) - F(s + nb)) - b \int_0^n P_1(t - a)F'(s + tb)dt. \tag{50}$$

Integrating the last integral by parts, using the first and third of (41) and the second part of (44), we see at once by complete induction that

$$\begin{aligned}\sum_{\nu=0}^{n-1} F(s + ab + \nu b) &= \int_0^n F(s + tb)dt + (\tfrac{1}{2} - a)(F(s) - F(s + nb)) \\ &+ \sum_{k=1}^{2m-1} (-1)^{[k(k+1)/2]+1} b^k P_{k+1}(a)(F^{(k)}(s + nb) - F^{(k)}(s)) \\ &+ (-1)^m b^{2m} \int_0^n P_{2m}(t - a)F^{(2m)}(s + tb)dt, \\ 0 \leqq a < 1, &\qquad b > 0.\end{aligned} \tag{51}$$

In the special case $a = 0$, the values of $P_{k+1}(a)$ are given by (48), and adding $F(s + nb)$ to both sides in (51), we obtain Euler's summation formula[7]

$$\begin{aligned}\sum_{\nu=0}^{n} F(s + \nu b) &= \int_0^n F(s + tb)dt + \tfrac{1}{2}(F(s + nb) + F(s)) \\ &+ \sum_{k=1}^{m} \frac{(-1)^{k-1}B_k}{(2k)!} b^{2k-1}(F^{(2k-1)}(s + nb) - F^{(2k-1)})(s)) \\ &+ (-1)^m b^{2m} \int_0^n P_{2m}(t)F^{(2m)}(s + tb)dt.\end{aligned} \tag{52}$$

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[7] Euler, L., Methodus generalis summandi progressiones. Comment. Acad. Petrop., vol. 6 (1732–33, published 1738), pp. 68–97, and independently, Maclaurin, C., A treatise on fluxions. Edinburgh, 1742, art. 352 and 828–30, 847. Neither author gives the remainder term. The proof in the text was given, in the special case $a = 0$, by Wirtinger, W., Einige Anwendungen der Euler-Maclaurinschen Summenformel, insbesondere auf eine Aufgabe von Abel. Acta Math., vol. 26 (1902), pp. 255–271.

In view of the application of (51) to the approximate evaluation of the sum to the left, it will be necessary to investigate the remainder term, or the last term to the right. The term corresponding to $k = 2m - 1$ in the sum to the right in (51) is

$$(-1)^{m-1}b^{2m-1}P_{2m}(a)(F^{(2m-1)}(s+nb) - F^{(2m-1)}(s)) = (-1)^{m-1}b^{2m}P_{2m}(a)\int_0^n F^{(2m)}(s+tb)\,dt,$$

so that (51) may be written in the form

$$\sum_{\nu=0}^{n-1} F(s+ab+\nu b) = \int_0^n F(s+tb)\,dt + (\tfrac{1}{2} - a)(F(s) - F(s+nb)) + \sum_{k=1}^{2m-2}(-1)^{[k(k+1)/2]+1}b^kP_{k+1}(a)(F^{(k)}(s+nb) - F^{(k)}(s)) + R_{2m}, \tag{53}$$

$$R_{2m} = (-1)^m b^{2m}\int_0^n (P_{2m}(t-a) - P_{2m}(a))F^{(2m)}(s+tb)\,dt. \tag{54}$$

Now suppose s and $F(s)$ *real*, and assume that *the sign of $F^{(2m)}(s+tb)$ does not change for $0 \leqq t \leqq n$*, so that, writing $\alpha = 1$ or -1, $\alpha F^{2m}(s+tb) \geqq 0$ in the interval of integration in (54). Since $P_{2m}(t)$ takes its maximum value for $t = 0$, we have

$$|P_{2m}(t-a) - P_{2m}(a)| \leqq |P_{2m}(t-a)| + |P_{2m}(a)| < 2P_{2m}(0),$$

and by (14) and (13),

$$\left|\int_0^n (P_{2m}(t-a) - P_{2m}(a))\cdot\alpha F^{(2m)}(s+tb)\,dt\right| \leqq \int_0^n 2P_{2m}(0)\cdot\alpha F^{(2m)}(s+tb)\,dt = \frac{2P_{2m}(0)}{b}\alpha(F^{(2m-1)}(s+nb) - F^{(2m-1)}(s)),$$

so that[8]

$$|R_{2m}| < 2b^{2m-1}P_{2m}(0)\,|F^{(2m-1)}(s+nb) - F^{(2m-1)}(s)|\,. \tag{55}$$

In the two most important special cases $a = 0$ and $a = \frac{1}{2}$, we shall now obtain more accurate approximate values of R_{2m} by introducing stronger assumptions regarding $F^{(2m)}(s)$.[9] Let $\alpha = \pm 1$, and assume *first, that*

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[8] For $a = 0$, a slightly more accurate formula was given by Poisson, S. D., "Sur le calcul numérique des Intégrales définies," Mém. Ac. Sc. Paris, vol. 6 (1827), pp. 571–602. Other investigations of the remainder term (in the special case $a = 0$) by Jacobi, C. G. J., "De uso legitimo formulæ summatoriæ Maclaurinianæ," Journ. f. Math., vol. 12 (1834), pp. 263–272, and Malmsten, C. J., "Sur la formule, etc.," Acta Math., vol. 5 (1884), pp. 1–46 (corrected reprint from Journ. f. Math., vol. 35 (1847), pp. 55–82).

[9] For $a = 0$, this method is due to Sonin, N., "Sur les termes complémentaires de la formule sommatoire d'Euler et de celle de Stirling," Ann. de l'Ec. Norm., ser. 3, vol. 6 (1889), pp. 257–262.

$\alpha F^{(2m)}(s + tb)$ *is positive and decreasing as t increases from* 0 *to* $n + \frac{1}{2}$, *and second, that*

$$\alpha \sum_{\nu=0}^{n-1} (F^{(2m)}(s + \nu b + tb) + F^{(2m)}(s + \nu b + b - tb))$$

is positive and decreasing as t increases from 0 to $\frac{1}{2}$. *Then, for* $a = 0$,

$$(-1)^{m-1}R_{2m} = b^{2m-1}P_{2m}(0)(F^{(2m-1)}(s + nb + hb) - F^{(2m-1)}(s + hb)), \qquad 0 < h < \tfrac{1}{2}. \tag{56}$$

Since $a = 0$, we may write (54) in the form

$$(-1)^{m-1}b^{-2m}\alpha R_{2m} = \int_0^n (P_{2m}(0) - P_{2m}(t))\alpha F^{(2m)}(s + tb)dt,$$

and since $P_{2m}(t)$ has the period 1,

$$(-1)^{m-1}b^{-2m}\alpha R_{2m} = \int_0^1 (P_{2m}(0) - P_{2m}(t))\alpha \sum_{\nu=0}^{n-1} F^{(2m)}(s + \nu b + tb)dt.$$

Decomposing the integral in two with limits 0, $\frac{1}{2}$ and $\frac{1}{2}$, 1 respectively, and introducing $1 - t$ as integration variable in the latter, we find by the aid of (44) that

$$(-1)^{m-1}b^{-2m}\alpha R_{2m} = \int_0^{\frac{1}{2}} (P_{2m}(0) - P_{2m}(t))\alpha \sum_{\nu=0}^{n-1} (F^{(2m)}(s + \nu b + tb) + F^{(2m)}(s + \nu b + b - tb)dt. \tag{57}$$

To obtain an upper bound for this expression, we shall use a theorem due to Tchebychef: *When* $u(t)$ *and* $v(t)$ *are continuous for* $a \leqq t \leqq b$, *and both these functions increase* (*or both decrease*) *as t increases, then*

$$(b - a)\int_a^b u(t)v(t)dt > \int_a^b u(t)dt \cdot \int_a^b v(t)dt,$$

but

$$(b - a)\int_a^b u(t)v(t)dt < \int_a^b u(t)dt \cdot \int_a^b v(t)dt$$

when one of the functions increases and the other decreases.[10] Writing

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[10] The following simple proof is due to Franklin, F., "Proof of a Theorem of Tchebycheff's on Definite Integrals," Am. Journ., vol. 7, pp. 377–379: Consider the double integral

$$\int_a^b\int_a^b (u(t) - u(x))(v(t) - v(x))dtdx = \int_a^b dt \int_a^b [u(t)v(t) - u(t)v(x) - u(x)v(t) + u(x)v(x)]dx$$

$$= (b - a)\int_a^b u(t)v(t)dt - \int_a^b u(t)dt \cdot \int_a^b v(x)dx - \int_a^b v(t)dt \cdot \int_a^b u(x)dx + (b - a)\int_a^b u(x)v(x)dx,$$

or replacing x by t in the single integrals to the right

$$\frac{1}{2}\int_a^b\int_a^b (u(t) - u(x))(v(t) - v(x))dtdx = (b - a)\int_a^b u(t)v(t)dt - \int_a^b u(t)dt \cdot \int_a^b v(t)dt.$$

When $u(t)$ and $v(t)$ vary in the same sense, $u(t) - u(x)$ and $v(t) - v(x)$ have the same sign, so

$u(t) = P_{2m}(0) - P_{2m}(t)$, we have, by (41), $u'(t) = P_{2m-1}(t) > 0$ for $0 < t < \frac{1}{2}$, so that $u(t)$ is positive and increases with t in $(0, \frac{1}{2})$, and by (41) and (43),

$$\int_0^{\frac{1}{2}} u(t)dt = [P_{2m}(0)\ t - P_{2m+1}(t)]_{t=0}^{t=\frac{1}{2}} = \tfrac{1}{2}P_{2m}(0).$$

The second factor in (57),

$$v(t) = \alpha \sum_{\nu=0}^{n-1} (F^{(2m)}(s + \nu b + tb) + F^{(2m)}(s + \nu b + b - tb)),$$

is positive and decreasing by hypothesis, and

$$\int_0^{\frac{1}{2}} v(t)dt = \int_0^n \alpha F^{(2m)}(s + tb)dt;$$

consequently, by Tchebychef's theorem,

$$P_{2m}(0) \int_0^n \alpha F^{(2m)}(s + tb)dt > (-1)^{m-1} b^{-2m} \alpha R_{2m}. \tag{58}$$

To obtain a lower bound for (57), we observe that $u(t)$ is positive and $v(t)$ decreasing in the interval of integration, whence

$$\int_0^{\frac{1}{2}} u(t)v(t)dt > v(\tfrac{1}{2}) \int_0^{\frac{1}{2}} u(t)dt = P_{2m}(0) \sum_{\nu=0}^{u-1} \alpha F^{(2m)}(s + \nu b + \tfrac{1}{2}b).$$

But $\alpha F^{(2m)}(s + \nu b + \frac{1}{2}b + tb)$ is positive and decreasing when t increases from 0 to 1, and consequently

$$\alpha F^{(2m)}(s + \nu b + \tfrac{1}{2}b) > \int_0^1 \alpha F^{(2m)}(s + \nu b + \tfrac{1}{2}b + tb)dt,$$

so that, by the preceding inequality,

$$(-1)^{m-1} b^{-2m} \alpha R_{2m} > P_{2m}(0) \int_0^n \alpha F^{(2m)}(s + \tfrac{1}{2}b + tb)dt. \tag{59}$$

Since by the first hypothesis on $F^{(2m)}(s)$, the integral

$$\int_0^n \alpha F^{(2m)}(s + hb + tb)dt$$

is a continuous and decreasing function of h as h increases from 0 to $\frac{1}{2}$, the comparison of (58) and (59) shows that there exists an h interior to

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that their product is positive, and therefore also the double integral, which proves the first part of the theorem. The second part follows from the first upon replacing $u(t)$ by $-u(t)$.

$(0, \frac{1}{2})$ such that

$$(-1)^{m-1}b^{-2m}\alpha R_{2m} = P_{2m}(0)\int_0^n \alpha F^{(2m)}(s + hb + tb)dt,$$

and evaluating the integral, we obtain (56).

Now let $\alpha = \pm 1$, and assume *first, that* $\alpha F^{(2m)}(s + tb)$ *is positive and decreasing as* t *increases from* 0 *to* $n + \frac{1}{2}$, *and second, that*

$$\alpha \sum_{\nu=0}^{n-1} (F^{(2m)}(s + \tfrac{1}{2}b + \nu b + tb) + F^{(2m)}(s + \tfrac{1}{2}b + \nu b - tb))$$

is positive and increasing as t *increases from* 0 *to* $\frac{1}{2}$. *Then, for* $a = \frac{1}{2}$,

$$(60) \qquad (-1)^{m-1}R_{2m} = b^{2m-1}P_{2m}(\tfrac{1}{2})(F^{(2m-1)}(s + nb + hb) - F^{(2m-1)}(s + hb)), \qquad 0 < h < \tfrac{1}{2}.$$

We obtain from (54), in the case $a = \frac{1}{2}$,

$$\begin{aligned}(-1)^m b^{-2m}R_{2m} &= \int_{-\frac{1}{2}}^{n-\frac{1}{2}} (P_{2m}(t) - P_{2m}(\tfrac{1}{2}))\cdot\alpha F^{(2m)}(s + \tfrac{1}{2}b + tb)dt \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} (P_{2m}(t) - P_{2m}(\tfrac{1}{2}))\alpha \sum_{\nu=0}^{n-1} F^{(2m)}(s + \tfrac{1}{2}b + \nu b + tb)dt\end{aligned}$$

and decomposing the integral in two with limits $-\frac{1}{2}$, 0 and 0, $\frac{1}{2}$ respectively, and introducing $-t$ as integration variable in the first, (44) gives

$$\begin{aligned}(-1)^m b^{-2m}R_{2m} = \int_0^{\frac{1}{2}} (P_{2m}(t) - P_{2m}(\tfrac{1}{2}))\cdot\alpha \sum_{\nu=0}^{n-1} (F^{(2m)}(s + \tfrac{1}{2}b + \nu b + tb) \\ + F^{(2m)}(s + \tfrac{1}{2}b + \nu b - tb))dt.\end{aligned}$$

Here the first factor under the integral sign is seen at once to be positive and decreasing, while the second factor is positive and increasing by hypothesis, and (60) is now readily proved in the same way as before.

CHAPTER II.

Definite integrals for the Gamma function derived from its definition as an infinite product.

10. Expression of the Gamma function by Euler's integral of the second kind. Integrals for $P(s)$ and $Q(s)$. Assume $\Re(s) > 0$ (when $s = x + yi$, $\Re(s) = x$ is the real part of s), let t be a real variable, and define t^s as $e^{s \log t}$, where $\log t$ denotes the real value of the logarithm of t; since

$$t^{s-1} = \frac{d}{dt}\left(\frac{t^s}{s}\right),$$

it follows that

$$\int_0^1 t^{s-1}dt = \lim_{\epsilon\to 0}\left[\frac{t^s}{s}\right]_{t=\epsilon}^{t=1} = \frac{1}{s}.$$

Replacing s by $s + 1$ and subtracting the new integral from the previous one, we find

$$\int_0^1 t^{s-1}(1 - t)dt = \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)},$$

and, generally, from

$$\int_0^1 t^{s-1}(1 - t)^{n-1}dt = \frac{(n-1)!}{s(s+1)\cdots(s+n-1)}$$

it follows that

$$\int_0^1 t^{s-1}(1 - t)^n dt = \frac{(n-1)!}{s(s+1)\cdots(s+n-1)} - \frac{(n-1)!}{(s+1)(s+2)\cdots(s+n)}$$

$$= \frac{n!}{s(s+1)\cdots(s+n)},$$

or finally, replacing t by t/n,

$$\int_0^n t^{s-1}\left(1 - \frac{t}{n}\right)^n dt = \frac{n!\,n^s}{s(s+1)\cdots(s+n)}.$$

For $n \to \infty$, the expression to the right approaches the limit $\Gamma(s)$, as is seen from ($J4''$) upon replacing n by $n + 1$ and a by unity, and since $(1 - t/n)^n \to e^{-t}$, the above equation suggests that

$$(61) \qquad \Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt, \qquad \Re(s) > 0.^{11}$$

To prove this, we must first establish the convergence of the infinite integral, which is known as Euler's integral of the second kind. Writing $s = x + yi$, and assuming $A \geqq x \geqq \epsilon > 0$, where A is as large and ϵ as small as we please, we have $|e^{-t}t^{s-1}| = e^{-t}t^{x-1}$, and for $0 < t \leqq 1$, $e^{-t}t^{x-1} < t^{\epsilon-1}$, while for t sufficiently large, $e^t > t^{A+1}$ and

$$e^{-t}t^{x-1} < t^{-(A+1)}t^{A-1} = t^{-2}.$$

Hence by theorem XVI corollary and theorem XIX, the integral in (61) exists uniformly for $A \geqq \Re(s) \geqq \epsilon > 0$. In the second place, it must be shown that the left and right members of (61) are equal, or that

$$\lim_{n\to\infty}\int_0^n t^{s-1}\left(1 - \frac{t}{n}\right)^n dt = \int_0^\infty e^{-t}t^{s-1}dt, \qquad \Re(s) > 0.$$

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[11] For s real and positive, this expression is due to Euler (J^{10}).

We have

$$\int_0^\infty e^{-t}t^{s-1}dt - \int_0^n t^{s-1}\left(1 - \frac{t}{n}\right)^n dt$$
$$= \int_0^\infty \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right)t^{s-1}dt + \int_n^\infty e^{-t}t^{s-1}dt,$$

and the second integral to the right approaches zero as $n \to \infty$, on account of the convergence of the integral in (61). Using the inequality[12]

$$0 \leqq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leqq 2\frac{t^2}{n}e^{-t} \qquad \text{for} \qquad 0 \leqq t \leqq n,$$

we find

$$\left|\int_0^n \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right)t^{s-1}dt\right| \leqq \int_0^n \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right)t^{x-1}dt$$
$$\leqq \int_0^n \frac{2t^2}{n} e^{-t}t^{x-1}dt < \frac{2}{n}\int_0^\infty e^{-t}t^{x+1}dt \to 0,$$

the last integral being convergent and independent of n. Thus the proof of (61) is complete.

Replacing t by at, where $a > 0$, we obtain from (61)

$$(62) \qquad \frac{\Gamma(s)}{a^s} = \int_0^\infty e^{-at}t^{s-1}dt, \qquad a > 0, \qquad \Re(s) > 0.$$

The integral

$$\int_0^\infty e^{-t}t^{s-1}\log t dt$$

converges uniformly for $A \geqq x \geqq \epsilon > 0$, since for t sufficiently small, $|e^{-t}t^{s-1}\log t| < t^{\epsilon-1}|\log t| < t^{(\epsilon/2)-1}$, and for t sufficiently large,

$$|e^{-t}t^{s-1}\log t| < t^{-2}\log t < t^{-3/2};$$

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[12] Proof: for $b \geqq a \geqq 0$, we have

$$0 \leqq b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1}) \leqq nb^{n-1}(b - a).$$

When $b = e^{-t/n}$, where $0 \leqq t \leqq n$, and $a = 1 - (t/n)$, we have

$$b - a = \frac{1}{2!}\left(\frac{t}{n}\right)^2 - \frac{1}{3!}\left(\frac{t}{n}\right)^3 + \cdots,$$

so that, the series being alternating with numerically decreasing terms,

$$0 \leqq b - a \leqq \frac{1}{2}\left(\frac{t}{n}\right)^2.$$

Consequently

$$0 \leqq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leqq ne^{-(1-1/n)t}\cdot\frac{1}{2}\left(\frac{t}{n}\right)^2 = \frac{e^{t/n}}{2}\cdot e^{-t}\frac{t^2}{n} \leqq \frac{e}{2}e^{-t}\frac{t^2}{n} \leqq 2e^{-t}\frac{t^2}{n}.$$

The inequality and its use in proving (61) are due to Watson, G. N., "An inequality associated with the Gamma function," Messenger of Math., vol. 45 (1915), pp. 28–30.

consequently by theorem XXI, we may differentiate (61) in respect to x under the integral sign, and since

$$\frac{\partial \Gamma(s)}{\partial x} = \frac{d\Gamma(s)}{ds},$$

we find

$$\frac{d\Gamma(s)}{ds} = \int_0^\infty e^{-t} t^{s-1} \log t dt, \qquad \Re(s) > 0, \tag{63}$$

and the repeated differentiation under the integral sign gives

$$\frac{d^n\Gamma(s)}{ds^n} = \int_0^\infty e^{-t} t^{s-1} (\log t)^n dt, \qquad \Re(s) > 0. \tag{64}$$

Integral expressions for the functions $P(s)$ and $Q(s)$ may be obtained in the following manner: the expansion

$$e^{-t} t^{s-1} = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} t^{s-1+\nu}$$

converges uniformly for $0 < \epsilon \leqq t \leqq 1$, and therefore, by theorem XXI corollary

$$\int_\epsilon^1 e^{-t} t^{s-1} dt = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \frac{1 - \epsilon^{s+\nu}}{s+\nu}.$$

For $\Re(s) = x > 0$, the integral $\int_0^1 e^{-t} t^{s-1} dt$ converges, and since

$$|\epsilon^{s+\nu}| \leqq \epsilon^x, \qquad |s+\nu| = |x+\nu+yi| \geqq x+\nu \geqq x,$$

we have

$$\left| \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \frac{\epsilon^{s+\nu}}{s+\nu} \right| \leqq \frac{\epsilon^x}{x} \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \rightarrow 0 \text{ as } \epsilon \rightarrow 0,$$

so that, by the definition of $P(s)$ (J33),

$$P(s) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!(s+\nu)} = \int_0^1 e^{-t} t^{s-1} dt, \qquad \Re(s) > 0. \tag{65}$$

Since $Q(s) = \Gamma(s) - P(s)$, (61) and (65) give

$$Q(s) = \int_1^\infty e^{-t} t^{s-1} dt. \tag{66}$$

As the integrand does not become infinite at the lower limit for any finite value of s, the last integral converges uniformly for $-A \leqq \Re(s) \leqq A$, where A is as large as we please, and consequently the entire function $Q(s)$ equals the integral (66) for all finite values of s, by theorem XXIII. The integrals (65) and (66) are due to Legendre and Prym ($J^{34,\,32}$).

11. The hypergeometric series $F(\alpha, \beta, \gamma;\, s)$ and its value for $s = 1$.

Euler's integral of the first kind. The name hypergeometric was applied by Wallis[13] to the power series in s

$$1+\frac{\alpha\cdot\beta}{1\cdot\gamma}s+\frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}s^2 + \frac{\alpha(\alpha+1)(\alpha+2)\cdot\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}s^3+\cdots,$$

where α, β and γ may have any real or complex values, except that γ must not equal zero or a negative integer (for when $\gamma = -n$, the $n+1$st and following coefficients become infinite). Using the notations

$$(67)\qquad \begin{aligned}(\alpha)_n &= \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1),\\ (\alpha, n) &= (n-1)!n^\alpha/(\alpha)_n, \qquad (\alpha, n)\to\Gamma(\alpha) \text{ as } n\to\infty,\end{aligned}$$

already introduced in J § 14 and J § 2 respectively, we may write this series

$$(68)\qquad F(\alpha, \beta, \gamma;\ s) = \sum_{\nu=0}^{\infty}\frac{(\alpha)_\nu(\beta)_\nu}{(1)_\nu(\gamma)_\nu}s^\nu.$$

Many of the elementary functions are special cases of this series; for instance

$$(1+s)^n = F(-n, \beta, \beta;\ s), \qquad \log(1+s) = sF(1, 1, 2;\ -s),$$
$$e^s = \lim_{\beta\to\infty}F(1, \beta, 1;\ s/\beta);$$

the essential properties of the hypergeometric series, and its intimate connection with the Gamma function, were discovered by Gauss (J^6). By (67), the absolute value of the general term in (68) may be written

$$\left|\frac{(1, \nu)(\gamma, \nu)}{(\alpha, \nu)(\beta, \nu)}\cdot\frac{s^\nu}{\nu^{1+\gamma-\alpha-\beta}}\right| = \left|\frac{(1, \nu)(\gamma, \nu)}{(\alpha, \nu)(\beta, \nu)}\right|\cdot\frac{|s|^\nu}{\nu^{1+\Re(\gamma-\alpha-\beta)}},$$

and since the first factor approaches

$$\left|\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\right|$$

as $\nu\to\infty$, and $\Sigma\ 1/\nu^{1+k}$ converges for $k>0$, but diverges for $k\leqq0$, it is evident that the series (68) converge absolutely for $|s|<1$, but is divergent for $|s|>1$, and that for $|s|=1$, the series is absolutely convergent when $\Re(\gamma-\alpha-\beta)>0$.[14] To find the value of $F(\alpha, \beta, \gamma;\ 1)$, which is a convergent series when $\Re(\gamma-\alpha-\beta)>0$, we begin by establishing the following relation between the three hypergeometric series $F(\alpha, \beta, \gamma; s)$, $F(\alpha, \beta, \gamma - 1; s)$ and $F(\alpha, \beta, \gamma + 1; s)$ for $|s| < 1$:

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[13] Wallis, J., Arithmetica infinitorum, Oxford, 1656.

[14] By more elaborate methods, it may be shown that for $|s| = 1$, the series is (conditionally) convergent for $0 \geqq R(\gamma-\alpha-\beta) > -1$, divergent for $R(\gamma-\alpha-\beta)\leqq -1$, except for $s=1$, where the series diverges for $R(\gamma-\alpha-\beta)\leqq 0$.

$$\begin{aligned}(69)\quad \gamma[\gamma - 1 - (2\gamma - \alpha - \beta - 1)s]F(\alpha, \beta, \gamma; s)& \\ + (\gamma - \alpha)(\gamma - \beta)sF(\alpha, \beta, \gamma + 1; s)& \\ - \gamma(\gamma - 1)(1 - s)F(\alpha, \beta, \gamma - 1; s) &= 0.\end{aligned}$$

In fact, the constant term on the left equals $\gamma(\gamma - 1) - \gamma(\gamma - 1) = 0$, and the coefficient of s^n, where $n > 0$, is seen by (68) to be

$$\begin{aligned}\gamma(\gamma - 1)\frac{(\alpha)_n(\beta)_n}{(1)_n(\gamma)_n} - \gamma(2\gamma - \alpha - \beta - 1)\frac{(\alpha)_{n-1}(\beta)_{n-1}}{(1)_{n-1}(\gamma)_{n-1}}& \\ + (\gamma - \alpha)(\gamma - \beta)\frac{(\alpha)_{n-1}(\beta)_{n-1}}{(1)_{n-1}(\gamma + 1)_{n-1}}& \\ - \gamma(\gamma - 1)\left(\frac{(\alpha)_n(\beta)_n}{(1)_n(\gamma - 1)_n} - \frac{(\alpha)_{n-1}(\beta)_{n-1}}{(1)_{n-1}(\gamma - 1)_{n-1}}\right)&\end{aligned}$$

or denoting by

$$u_n = \frac{(\alpha)_n(\beta)_n}{(1)_n(\gamma - 1)_n}$$

the coefficient of s^n in $F(\alpha, \beta, \gamma - 1; s)$,

$$\begin{aligned}\frac{\gamma(\gamma - 1)u_n}{\gamma + n - 1}\Bigg(\gamma - 1 + \frac{n(1 + \alpha + \beta - 2\gamma)(\gamma + n - 1)}{(\alpha + n - 1)(\beta + n - 1)}& \\ + \frac{n(\gamma - \alpha)(\gamma - \beta)}{(\alpha + n - 1)(\beta + n - 1)} - (\gamma + n - 1)& \\ + \frac{n(\gamma + n - 2)(\gamma + n - 1)}{(\alpha + n - 1)(\beta + n - 1)}\Bigg)& \\ = \frac{n\gamma(\gamma - 1)u_n}{\gamma + n - 1}\Bigg(-1 + \frac{(\gamma - \alpha)(\gamma - \beta)}{(\alpha + n - 1)(\beta + n - 1)}& \\ + \frac{(\alpha + \beta - \gamma + n - 1)(\gamma + n - 1)}{(\alpha + n - 1)(\beta + n - 1)}\Bigg) &= 0.\end{aligned}$$

In (69) we now let s tend toward unity through positive and increasing values; since $F(\alpha, \beta, \gamma; 1)$ and $F(\alpha, \beta, \gamma + 1; 1)$ are convergent series on account of $\Re(\gamma - \alpha - \beta) > 0$, Abel's theorem on power series (J^{31}) shows that

$$\lim_{s\to 1} F(\alpha, \beta, \gamma; s) = F(\alpha, \beta, \gamma; 1)$$

and

$$\lim_{s\to 1} F(\alpha, \beta, \gamma + 1; s) = F(\alpha, \beta, \gamma + 1; 1).$$

Furthermore $(1 - s)F(\alpha, \beta, \gamma - 1; s) = 1 + \sum_1^\infty (u_\nu - u_{\nu-1})s^\nu$ for $|s| < 1$

and since

$$1 + \sum_{1}^{\infty} (u_\nu - u_{\nu-1}) = \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{(1, n)(\gamma - 1, n)}{(\alpha, n)(\beta, n)} \cdot \frac{1}{n^{\gamma-\alpha-\beta}} = 0$$

on account of $\Re(\gamma - \alpha - \beta) > 0$, the theorem in question gives

$$\lim_{s\to 1} (1 - s)F(\alpha, \beta, \gamma - 1; s) = 0.$$

Thus we finally obtain from (69)

$$F(\alpha, \beta, \gamma; 1) = \frac{(\gamma - \alpha)(\gamma - \beta)}{\gamma(\gamma - \alpha - \beta)} F(\alpha, \beta, \gamma + 1; 1)$$

and replacing γ by $\gamma + 1, \gamma + 2, \cdots, \gamma + n - 1$ and multiplying the corresponding equations

$$\begin{aligned} F(\alpha, \beta, \gamma; 1) &= \frac{(\gamma - \alpha)_n(\gamma - \beta)_n}{(\gamma)_n(\gamma - \alpha - \beta)_n} F(\alpha, \beta, \gamma + n; 1) \\ &= \frac{(\gamma, n)(\gamma - \alpha - \beta, n)}{(\gamma - \alpha, n)(\gamma - \beta, n)} F(\alpha, \beta, \gamma + n; 1). \end{aligned}$$

The first factor to the right approaches the limit $\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)/\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)$ as $n \to \infty$, and for $n > |\gamma|$, we have

$$\begin{aligned} |F(\alpha, \beta, \gamma + n; 1) - 1| &= \left| \sum_{\nu=1}^{\infty} \frac{(1)_\nu(\gamma + n)_\nu}{(\alpha)_\nu(\beta)_\nu} \right| \\ &\leqq \sum_{\nu=1}^{\infty} \left| \frac{(\alpha)_\nu(\beta)_\nu}{(1)_\nu(\gamma + n)_\nu} \right| \leqq \sum_{\nu=1}^{\infty} \frac{(|\alpha|)_\nu(|\beta|)_\nu}{(1)_\nu(n - |\gamma|)_\nu} \\ &< \frac{|\alpha|\,|\beta|}{n - |\gamma|} \sum_{\nu=1}^{\infty} \frac{(|\alpha| + 1)_\nu(|\beta| + 1)_\nu}{(1)_\nu(n + 1 - |\gamma|)_\nu}; \end{aligned}$$

the last series converges for $n > |\gamma| + |\alpha| + |\beta| - 1$ and is evidently a decreasing function of n, and since $1/(n - |\gamma|) \to 0$ as $n \to \infty$, we have $\lim_{n\to\infty} F(\alpha, \beta, \gamma + n; 1) = 1$. We therefore obtain the final result

$$(70) \quad F(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}, \qquad \Re(\gamma - \alpha - \beta) > 0.$$

From this formula we shall now derive the relation

$$(71) \quad \int_0^1 t^{\alpha-1}(1 - t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \qquad \Re(\alpha) > 0, \qquad \Re(\beta) > 0,$$

where the integral is known as Euler's integral of the first kind[15] Multiplying the binomial expansion, convergent for $|t| < 1$,

---

[15] J^{10} (for α and β real and positive).

$$(1 - t)^{\beta-1} = \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{(\beta - 1)(\beta - 2)\cdots(\beta - \nu)}{\nu!} t^{\nu} = \sum_{\nu=0}^{\infty} \frac{(\alpha)_{\nu}(1 - \beta)_{\nu}}{(1)_{\nu}(\alpha)_{\nu}} t^{\nu}$$

by $t^{\alpha-1}$ and integrating from ϵ to s $(0 < \epsilon < s < 1)$, we obtain, since the series converges uniformly

$$\int_{\epsilon}^{s} t^{\alpha-1}(1 - t)^{\beta-1}dt = \sum_{\nu=0}^{\infty} \frac{(\alpha)_{\nu}(1 - \beta)_{\nu}}{(1)_{\nu}(\alpha)_{\nu}(\alpha + \nu)} (s^{\alpha+\nu} - \epsilon^{\alpha+\nu})$$

$$= \frac{s^{\alpha}}{\alpha} F(\alpha, 1 - \beta, \alpha + 1; s) - \frac{\epsilon^{\alpha}}{\alpha} F(\alpha, 1 - \beta, \alpha + 1; \epsilon).$$

Now let $\epsilon \to 0$ and $s \to 1$; since $\mathfrak{R}(\alpha) > 0$, $\mathfrak{R}(\beta) > 0$, the integral tends toward the left member of (71), $\epsilon^{\alpha} \to 0$ and on account of

$$\mathfrak{R}(\alpha + 1 - \alpha - (1 - \beta)) = \mathfrak{R}(\beta) > 0,$$

$F(\alpha, 1 - \beta, \alpha + 1; 1)$ is convergent, so that Abel's theorem on power series gives $F(\alpha, 1 - \beta, \alpha + 1; s) \to F(\alpha, 1 - \beta, \alpha + 1; 1)$. Applying (70), we therefore obtain

$$\int_{0}^{1} t^{\alpha-1}(1 - t)^{\beta-1}dt = \frac{1}{\alpha} F(\alpha, 1 - \beta, \alpha + 1; 1) = \frac{1}{\alpha} \frac{\Gamma(\alpha + 1)\Gamma(\beta)}{\Gamma(1)\Gamma(\alpha + \beta)}$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

We may now express the hypergeometric series as a definite integral. Let s have a fixed value such that $| s | < 1$, then $| st | \leqq | s |$ for $0 \leqq t \leqq 1$, and the binomial expansion

$$(1 - st)^{-\beta} = \sum_{\nu=0}^{\infty} \frac{(\beta)_{\nu}}{(1)_{\nu}} s^{\nu}t^{\nu}$$

converges uniformly in t for $0 \leqq t \leqq 1$. Assuming $\mathfrak{R}(\alpha) > 0$, $\mathfrak{R}(\gamma - \alpha) > 0$, multiplying by $t^{\alpha-1}(1 - t)^{\gamma-\alpha-1}$ and integrating from 0 to 1, we may therefore integrate term by term on the right side and obtain

$$\int_{0}^{1} t^{\alpha-1}(1 - t)^{\gamma-\alpha-1}(1 - st)^{-\beta}dt = \sum_{\nu=0}^{\infty} \frac{(\beta)_{\nu}}{(1)_{\nu}} s^{\nu} \int_{0}^{1} t^{\alpha+\nu-1}(1 - t)^{\gamma-\alpha-1}dt$$

$$= \sum_{\nu=0}^{\infty} \frac{(\beta)_{\nu}}{(1)_{\nu}} s^{\nu} \cdot \frac{\Gamma(\alpha + \nu)\Gamma(\gamma - \alpha)}{\Gamma(\gamma + \nu)}$$

by (71), or since $\Gamma(\alpha + \nu) = (\alpha)_{\nu}\Gamma(\alpha)$ by $(J2)$,

$$(72) \quad \int_{0}^{1} t^{\alpha-1}(1 - t)^{\gamma-\alpha-1}(1 - st)^{-\beta}dt = \frac{\Gamma(\alpha)\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} F(\alpha, \beta, \gamma; s),$$

$$| s | < 1, \quad \mathfrak{R}(\alpha) > 0, \quad \mathfrak{R}(\gamma - \alpha) > 0.$$

12. **Application of Euler's summation formula to** $\log \Gamma(s+a)$ **and** $\psi(s+a)$. To secure the single-valuedness of the logarithms with which we shall deal in the following, we introduce a cut in the complex plane along the negative real axis (from 0 to $-\infty$) and subject the complex variable s to the restriction that it shall never cross this cut. Then every branch of the function $\log s$ is single-valued, and we shall define $\log s$ as that particular branch which is real for real and positive values of s. In particular, we have $\log (s)_n = \log s + \log (s+1) + \cdots + \log (s+n-1)$. We shall now apply (50) and (51) to the case $F(s) = \log s$, $b = 1$ and obtain first a proof that (s, n) approaches a limit as $n \rightarrow \infty$ for any s not on the negative real axis (that is, a new proof of the convergence of Euler's infinite product for $\Gamma(s)$, J3 and 3′, for these values of s) and second, Stirling's asymptotic series for $\log \Gamma(s)$.

Making $F(s) = \log s$, $b = 1$, $0 \leqq a < 1$ in (50), we find, since $\int_0^n \log (s+t)dt = (s+n)(\log (s+n) - 1) - s(\log s - 1)$ and $F'(s) = \frac{1}{s}$,

$$\log (s+a)_n = \sum_{\nu=0}^{n-1} \log (s+a+\nu) = (n+s+a-\tfrac{1}{2}) \log (s+n)$$
$$- n - (s+a-\tfrac{1}{2}) \log s - \int_0^n \frac{P_1(t-a)}{s+t} dt;$$

in particular, we find for $s = 1$, $a = 0$ and replacing n by $n-1$

$$\log (n-1)! = \sum_{\nu=0}^{n-2} \log (1+\nu) = (n-\tfrac{1}{2}) \log n - n + 1 - \int_0^{n-1} \frac{P_1(t)}{1+t} dt.$$

Adding $(s+a) \log n$ to the last equation, and subtracting the first, we obtain

$$\log (s+a, n) = \log (n-1)! + (s+a) \log n - \log (s+a)_n$$
$$= -(n+s+a-\tfrac{1}{2}) \log \left(1 + \frac{s}{n}\right) + (s+a-\tfrac{1}{2}) \log s \tag{73}$$
$$+ 1 - \int_0^{n-1} \frac{P_1(t)}{1+t} dt + \int_0^n \frac{P_1(t-a)}{s+t} dt.$$

Since $P_1(t) = -\frac{dP_2(t)}{dt}$ by (41), an integration by parts gives

$$\int_0^n \frac{P_1(t-a)}{s+t} dt = \frac{P_2(a)}{s} - \frac{P_2(a)}{s+n} - \int_0^n \frac{P_2(t-a)}{(s+t)^2} dt,$$

and as $| P_2(t-a) | \leqq P_2(0)$ by (42), the integral $\int_0^\infty \frac{P_2(t-a)}{(s+t)^2} dt$ converges for every s not on the negative real axis (theor. XVI cor.). Consequently

$$\lim_{n\to\infty}\int_0^n \frac{P_1(t-a)}{s+t}\,dt = \int_0^\infty \frac{P_1(t-a)}{s+t}\,dt = \frac{P_2(a)}{s} - \int_0^\infty \frac{P_2(t-a)}{(s+t)^2}\,dt, \tag{74}$$

and similarly $\lim\limits_{n\to\infty}\int_0^{n-1}\frac{P_1(t)}{1+t}\,dt = \int_0^\infty \frac{P_1(t)}{1+t}\,dt$ exists; furthermore $\lim\limits_{n\to\infty}(n+s+a-\tfrac{1}{2})\log\left(1+\frac{s}{n}\right) = \lim\limits_{n\to\infty}(n+s+a-\tfrac{1}{2})\cdot\frac{s}{n} = s$, and substituting all this in (73) we find

$$\lim_{n\to\infty}\log(s+a,n) = (s+a-\tfrac{1}{2})\log s - s + 1 - \int_0^\infty \frac{P_1(t)}{1+t}\,dt + \int_0^\infty \frac{P_1(t-a)}{s+t}\,dt.$$

From the existence of the limit to the left we infer at once that $\lim\limits_{n\to\infty}(s+a,n) = \Gamma(s+a)$ exist for every s not on the cut, and that for such values of s

$$\log\Gamma(s+a) = (s+a-\tfrac{1}{2})\log s - s + k' + \int_0^\infty \frac{P_1(t-a)}{s+t}\,dt, \tag{75}$$

where we now only have to evaluate the constant $k' = 1 - \int_0^\infty \frac{P_1(t)}{1+t}\,dt$. To this purpose we observe that $\int_0^\infty \frac{P_1(t)}{n+t}\,dt = \int_{n-1}^\infty \frac{P_1(t)}{1+t}\,dt \to 0$ as $n\to\infty$ on account of the convergence of $\int_0^\infty \frac{P_1(t)}{1+t}\,dt$; making $a = 0$, $s = n$ and $s = 2n$ in (75), we obtain

$$2\log\Gamma(n) = (2n-1)\log n - 2n + 2k' + 2\int_0^\infty \frac{P_1(t)}{n+t}\,dt,$$

$$\log\Gamma(2n) = (2n-\tfrac{1}{2})\log n - 2n + k' + \int_0^\infty \frac{P_1(t)}{2n+t}\,dt,$$

and subtracting

$$k' + 2\int_0^\infty \frac{P_1(t)}{n+t}\,dt - \int_0^\infty \frac{P_1(t)}{2n+t}\,dt = \log\frac{\Gamma(n)^2 n^{\frac{1}{2}} 2^{2n-\frac{1}{2}}}{\Gamma(2n)}$$

$$= \log\frac{2^{\frac{1}{2}}\Gamma(n)n^{\frac{1}{2}}}{\frac{1}{2}(\frac{1}{2}+1)\cdots(\frac{1}{2}+n-1)},$$

so that, as $n\to\infty$, $k' = \log(2^{\frac{1}{2}}\Gamma(\tfrac{1}{2})) = \log\sqrt{2\pi}$. This is essentially the method given in J § 4; another is the following: making $s = \frac{1}{2} \pm yi$, $y > 0$ and $a = 0$, so that for $t \geqq 0$, $|s+t| > y$ and $|s+t| > \frac{1}{2}+t$, $|s+t|^2 > y^{\frac{1}{2}}(\frac{1}{2}+t)^{\frac{3}{2}}$, it follows from (74) that

$$\left|\int_0^\infty \frac{P_1(t)}{\frac{1}{2} \pm yi + t}dt\right| < \frac{P_2(0)}{y} + \frac{P_2(0)}{y^{\frac{1}{2}}}\int_0^\infty \frac{dt}{(\frac{1}{2}+t)^{\frac{3}{2}}} \to 0 \text{ as } y \to \infty.$$

Now, by ($J6$)

$$\Gamma(\tfrac{1}{2} + yi)\Gamma(\tfrac{1}{2} - yi) = \frac{\pi}{\sin \pi(\frac{1}{2} + yi)} = \frac{2\pi i}{e^{\pi i(\frac{1}{2}+yi)} - e^{-\pi i(\frac{1}{2}+yi)}}$$

$$= \frac{2\pi}{e^{-\pi y} + e^{\pi y}} = \frac{2\pi e^{-\pi y}}{1 + e^{-2\pi y}},$$

and adding the two equations (75) for $a = 0$, $s = \frac{1}{2} + yi$ and $s = \frac{1}{2} - yi$,

$$\log 2\pi - \pi y - \log(1 + e^{-2\pi y})$$

$$= yi \log(\tfrac{1}{2} + yi) - (\tfrac{1}{2} + yi) - yi \log(\tfrac{1}{2} - yi) - (\tfrac{1}{2} - yi)$$

$$+ 2k' + \int_0^\infty \frac{P_1(t)}{\frac{1}{2} + yi + t}dt + \int_0^\infty \frac{P_1(t)}{\frac{1}{2} - yi + t}dt.$$

But $\log(\frac{1}{2} + yi) = \log y + \log i + \log\left(1 + \frac{1}{2yi}\right) = \log y + \frac{\pi i}{2} + \frac{1}{2yi} + \cdots$, and similarly $\log(\frac{1}{2} - yi) = \log y - \frac{\pi i}{2} - \frac{1}{2yi} + \cdots$; introducing this in the preceding equation and letting $y \to \infty$, we find $2k' = \log 2\pi$. Thus (75) finally becomes[16]

$$\log \Gamma(s + a) = (s + a - \tfrac{1}{2}) \log s - s + \log \sqrt{2\pi} + \omega(s, a),$$

(76)
$$\omega(s, a) = \int_0^\infty \frac{P_1(t - a)}{s + t}dt.$$

When $a = 0$,

$$\omega(s, 0) = \int_0^\infty \frac{P_1(t)}{s + t}dt = \sum_{\nu=0}^{\infty}\int_\nu^{\nu+1} \frac{P_1(t)}{s + t}dt$$

$$= \sum_{\nu=0}^{\infty}\int_0^1 \frac{P_1(t)}{s + \nu + t}dt = \sum_{\nu=0}^{\infty}\int_0^1 \frac{\frac{1}{2} - t}{s + \nu + t}dt,$$

and performing the integrations in the last expression, we obtain Gudermann's formula for $\omega(s, 0)$, (J 13).

In (76), we now integrate the expression for $\omega(s, a)$ by parts; the result may be read off immediately from a comparison of (50) and (51), observing that $F^k(s) = (-1)^{k-1}(k - 1)!/s^k$. Thus we obtain

---

[16] For $a = 0$ and s real, this formula is due to Gilbert, Ph. (J^{22}), and for $a = 0$ and s complex to Stieltjes, T. J., Sur le développement de log $\Gamma(a)$, Journ. de Math., ser. 4, vol. 5 (1889), pp. 425–444. For $0 \leqq a < 1$ and s complex, see Lindelöf, E., Le calcul des résidus, Paris, 1905, pp. 95–97.

$$(77)\quad \omega(s, a) = \sum_{k=1}^{2m-1} (-1)^{[k(k-1)]/2} \frac{(k-1)!P_{k+1}(a)}{s^k} + (-1)^m(2m-1)! \int_0^\infty \frac{P_{2m}(t-a)}{(s+t)^{2m}}\,dt.$$

To form an estimate of the remainder, let $s = |s|e^{\theta i}$, $-\pi < \theta < \pi$; then

$$|s+t|^2 = (|s|\cos\theta + t)^2 + |s|^2 \sin^2\theta$$

$$= (|s| + t)^2 \cos^2\frac{\theta}{2} + (|s| - t)^2 \sin^2\frac{\theta}{2} \geqq (|s| + t)^2 \cos^2\frac{\theta}{2},$$

and since $|P_{2m}(t-a)| \leqq P_{2m}(0)$, we obtain

$$\left|(-1)^m(2m-1)!\int_0^\infty \frac{P_{2m}(t-a)}{(s+t)^{2m}}\,dt\right| < \frac{(2m-1)!P_{2m}(0)}{\cos^{2m}\theta/2}\int_0^\infty \frac{dt}{(|s|+t)^{2m}}$$

$$= \frac{(2m-2)!P_{2m}(0)}{\cos^{2m}\theta/2\cdot|s|^{2m-1}}.$$

Joining the term with $k = 2m - 1$ in (77) to the remainder, we may therefore write

$$(78)\quad \omega(s, a) = \sum_{k=1}^{2m-2} (-1)^{[k(k-1)]/2} \frac{(k-1)!P_{k+1}(a)}{s^k} + \frac{(2m-2)!P_{2m}(0)\,h}{\cos^{2m}\theta/2\cdot|s|^{2m-1}}, \qquad |h| < 2.$$

In the particular cases where $a = 0$ or $a = \frac{1}{2}$, $P_{k+1}(a) = 0$ for $k + 1$ odd, and therefore, replacing k by $2k - 1$, (78) becomes

$$(79)\quad \omega(s, a) = \sum_{k=1}^{m-1} (-1)^{k-1} \frac{(2k-2)!P_{2k}(a)}{s^{2k+1}} + \frac{(2m-2)!P_{2m}(0)\cdot h}{\cos^{2m}\theta/2\cdot|s|^{2m-1}}, \qquad |h| < 2, \quad a = 0 \text{ or } \tfrac{1}{2}.$$

When s is real and positive, we may use the much more accurate remainder terms (56) and (60). In fact, for $a = 0$ and $\alpha = -1$,

$$\alpha F^{(2m)}(s+t) = \frac{(2m-1)!}{(s+t)^{2m}}$$

is positive and decreases as t increases from 0 to ∞, and

$$\alpha(F^{(2m)}(s+\nu+t) + F^{(2m)}(s+\nu+1-t))$$

$$= \frac{(2m-1)!}{(s+\nu+t)^{2m}} + \frac{(2m-1)!}{(s+\nu+1-t)^{2m}}$$

is positive and decreases as t increases from 0 to $\frac{1}{2}$ (since the derivative in respect to t is negative). Similarly we see that for $a = \frac{1}{2}$ and $\alpha = -1$, the conditions for the applicability of (60) are satisfied. Taking the values of $P_{2k}(0)$ and $P_{2k}(\frac{1}{2})$ from (48), we therefore obtain[17]

$$\omega(s,0) = \sum_{k=1}^{m-1} \frac{(-1)^{k-1}B_k}{(2k-1)\cdot 2k} \cdot \frac{1}{s^{2k-1}} + \frac{(-1)^{m-1}B_m}{(2m-1)\cdot 2m}\,\frac{1}{(s+h)^{2m-1}}, \tag{80}$$

$$0 < h < \tfrac{1}{2}, \qquad s > 0.$$

$$\begin{aligned}\omega(s,\tfrac{1}{2}) &= \sum_{k=1}^{m-1}\left(1 - \frac{1}{2^{2k-1}}\right)\frac{(-1)^k B_k}{(2k-1)\cdot 2k} \cdot \frac{1}{s^{2k-1}} \\ &+ \left(1 - \frac{1}{2^{2m-1}}\right)\frac{(-1)^m B_m}{(2m-1)\cdot 2m}\,\frac{1}{(s+h)^{2m-1}}, \quad 0<h<\tfrac{1}{2}, \quad s>0.\end{aligned} \tag{81}$$

The infinite series

$$\sum_{k=1}^{\infty} (-1)^{[k(k-1)]/2}\,\frac{(k-1)!P_{k+1}(a)}{s^k},$$

the first $2m - 2$ terms of which occur in (78), does not converge for any value of s. First suppose $a \neq 0$ and $a \neq \frac{1}{2}$, then the ratio of the terms for which $k = 2m + 2$ and $k = 2m$ may be written, using the expressions (42) for $P_{2m+3}(a)$ and $P_{2m+1}(a)$:

$$-\frac{m(2m+1)}{2\pi^2 s^2} \cdot \frac{\displaystyle\sum_1^{\infty} \frac{\sin 2n\pi a}{n^{2m+2}}}{\displaystyle\sum_1^{\infty} \frac{\sin 2n\pi a}{n^{2m}}},$$

and since $\sum_1^{\infty} \frac{\sin 2n\pi a}{n^{2m}}$ converges uniformly in respect to m for $m \geqq 1$, we have $\sum_1^{\infty} \frac{\sin 2n\pi a}{n^{2m}} \to \sin 2\pi a$ as $m \to \infty$, so that the second factor in our ratio approaches unity as $m \to \infty$. The first factor, however, approaches infinity, and consequently the series cannot converge. When $a = 0$ or $a = \frac{1}{2}$, a similar argument applies to the ratio of the terms with $k = 2m + 3$ and $k = 2m + 1$.

Proceeding to the logarithmic derivative of the Gamma function, we find, making $F(s) = \frac{1}{s}$, $b = 1$, $0 \leqq a < 1$ in (50), that

$$\sum_{\nu=0}^{n-1} \frac{1}{s+a+\nu} = \log(s+n) - \log s + (\tfrac{1}{2} - a)\left(\frac{1}{s} - \frac{1}{s+n}\right) + \int_0^n \frac{P_1(t-a)\,dt}{(s+t)^2},$$

and since the last integral converges as $n \to \infty$, and $\psi(s+a) = \lim_{n\to\infty}\left(\log n - \sum_{\nu=0}^{n-1} \frac{1}{s+a+\nu}\right)$ by (J 19′), it is seen that

---

[17] (80) is due to Stirling (J^{21}) and (81) to Gauss (J^{11}), both without remainder terms. The form of the remainder term given in (80) is due to Sonin.[9] For $a = 0$, the remainder term in (79) was given by Stieltjes[16] (who in this particular case obtains $| h | \leqq 1$).

$$
\begin{aligned}
\psi(s+a) &= \log s - \omega^*(s, a) \\
\omega^*(s, a) &= (\tfrac{1}{2} - a)\frac{1}{s} + \int_0^\infty \frac{P_1(t-a)dt}{(s+t)^2}\,.
\end{aligned} \tag{82}
$$

Treating the integral in the same manner as the corresponding one for $\omega(s, a)$, we readily obtain

$$
\begin{aligned}
\omega^*(s, a) = (\tfrac{1}{2} - a)\frac{1}{s} &+ \sum_{k=2}^{2m} (-1)^{[(k-1)(k-2)]/2} \frac{(k-1)!P_k(a)}{s^k} \\
&+ (-1)^m (2m)! \int_0^\infty \frac{P_{2m}(t-a)dt}{(s+t)^{2m+1}}\,,
\end{aligned} \tag{83}
$$

$$
\begin{aligned}
\omega^*(s, a) = (\tfrac{1}{2} - a)\frac{1}{s} &+ \sum_{k=2}^{2m-1} (-1)^{[(k-1)(k-2)]/2} \frac{(k-1)!P_k(a)}{s^k} \\
&+ \frac{(2m-1)!P_{2m}(0)\cdot h}{\cos^{2m}\theta/2\cdot|s|^{2m}}\,, \qquad |h| < 2,
\end{aligned} \tag{84}
$$

and for s real and positive

$$
\omega^*(s, 0) = \frac{1}{2s} + \sum_{k=1}^{m-1} \frac{(-1)^{k-1}B_k}{2k} \cdot \frac{1}{s^{2k}} + \frac{(-1)^{m-1}B_m}{2m} \cdot \frac{1}{(s+h)^{2m}}\,, \qquad 0 < h < \tfrac{1}{2}, \tag{85}
$$

$$
\begin{aligned}
\omega^*(s, \tfrac{1}{2}) = \sum_{k=1}^{m-1} &\left(1 - \frac{1}{2^{2k-1}}\right) \frac{(-1)^k B_k}{2k} \cdot \frac{1}{s^{2k}} \\
&+ \left(1 - \frac{1}{2^{2m-1}}\right) \frac{(-1)^m B_m}{2m} \cdot \frac{1}{(s+h)^{2m}}\,, \qquad 0 < h < \tfrac{1}{2};
\end{aligned} \tag{86}
$$

furthermore, the same argument as before shows the divergence of the infinite series of which the initial terms occur in (84).

A divergent series

$$
a_0 + \frac{a_1}{s} + \frac{a_2}{s^2} + \cdots + \frac{a_n}{s^n} + \cdots,
$$

in which the sum of the $n + 1$ first terms is $S_n(s)$, is said to be an asymptotic expansion of a function $f(s)$ for a given range of values of $\theta = \text{arc } s$, if the expansion

$$
R_n(s) = s^n(f(s) - S_n(s))
$$

satisfies the condition $\lim\limits_{|s|\to\infty} R_n(s) = 0$ for n fixed, even though $\lim\limits_{n\to\infty} |R_n(s)| = \infty$ when s is fixed. In this sense, (78) and (79) are asymptotic expansions of $\omega(s, a)$, and (84) of $\omega^*(s, a)$, for $-\pi + \epsilon \leqq \theta \leqq \pi - \epsilon$, where ϵ is as small as we please, i. e., for s inside of any infinite sector not containing the negative real axis.

It is clear from the definition that a function $f(s)$ cannot have two different asymptotic expansions $a_0 + \frac{a_1}{s} + \cdots$ and $b_0 + \frac{b_1}{s} + \cdots$ both valid in the same sector (or even for a single common value of $\theta = \text{arc } s$), since then we should have

$$\lim_{|s| \to \infty} s^n \left(a_0 - b_0 + \frac{a_1 - b_1}{s} + \cdots + \frac{a_n - b_n}{s^n} \right) = 0,$$

whence $a_0 = b_0$, $a_1 = b_1$, $\cdots$. This remark will be used in the following to identify various expressions for the coefficients in (78) and (84).

13. Integrals for $\log \Gamma(s)$ **and** $\psi(s)$ **of Cauchy and Gauss.** We begin by showing that

$$(87) \qquad \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log \frac{b}{a} \text{ for } \Re(a) > 0, \Re(b) > 0,$$

where $\log \frac{b}{a}$ is the principal logarithm. By direct integration we find, for $x \geqq \epsilon > 0$,

$$\int_0^T e^{-(x+iy)t} dt = \frac{1}{x+iy}(1 - e^{-(x+iy)T}),$$

and since $\left| \frac{e^{-(x+iy)T}}{x+iy} \right| \leqq \frac{e^{-xT}}{x} \leqq \frac{e^{-\epsilon T}}{\epsilon} \to 0$ as $T \to \infty$, it follows that

$$(88) \qquad \int_0^\infty e^{-(x+iy)t} dt = \frac{1}{x+iy},$$

the integral being uniformly convergent for $x \geqq \epsilon$. Let $\Re(a) \geqq \epsilon$, $\Re(b) \geqq \epsilon$, and write $x + iy = a(1 - u) + bu$ where $0 \leqq u \leqq 1$, then $x \geqq \epsilon$, and from the uniform convergence of our integral it follows, by theor. XX, that

$$\int_0^1 \frac{(b-a)du}{a + (b-a)u} = \int_0^1 du \int_0^\infty (b - a) e^{-(a+(b-a)u)t} dt$$

$$= \int_0^\infty dt \int_0^1 (b - a) e^{-(a+(b-a)u)t} du$$

$$= \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt.$$

But the first single integral is found by direct integration to equal one of the values of $\log \frac{b}{a}$, and since the real part of the denominator in the integrand is not less than ϵ for $\Re(a) \geqq \epsilon$, $\Re(b) \geqq \epsilon$, it follows that the integral is continuous for such values of a and b. Since the integral is obviously real for real values of a and b, it follows that the value of $\log \frac{a}{b}$ in question must be the principal one, and (87) is proved. In the equation (J 19″), making $a = 1$,

$$\psi(s) = \lim_{n\to\infty}\left(\log (n+1) - \sum_{\nu=0}^{n-1}\frac{1}{s+\nu}\right)$$

we now assume $\Re(s) > 0$; then we have, by (88) and (87), for $\nu = 0, 1, \cdots,$

$$\frac{1}{s+\nu} = \int_0^\infty e^{-(s+\nu)t}dt, \qquad \log (n+1) = \int_0^\infty \frac{e^{-t} - e^{-(n+1)t}}{t}\,dt,$$

and consequently

$$\psi(s) = \lim_{n\to\infty}\int_0^\infty\left(\frac{e^{-t} - e^{-(n+1)t}}{t} - \sum_{\nu=0}^{n-1} e^{-(s+\nu)t}\right)dt.$$

Upon summation of the finite geometric series to the right, this becomes

$$\psi(s) = \lim_{n\to\infty}\int_0^\infty\left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}}\right)(1 - e^{-nt})dt.$$

Now let $\Re(s) \geqq \epsilon$ and $|\,s\,| < A$, where $\epsilon < 1$ is as small and A as large as we please; using the power series for e^{-t} and e^{-st}, we see that the first factor under the integral sign is expansible in a power series in t, convergent for $|\,t\,| < r$, where r is independent of s, and consequently

$$\left|\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}}\right| < M_1 = M_1(\epsilon, A)$$

for $0 \leqq t < r$. On the other hand, we have for $t \geqq r$, since $\epsilon < 1$

$$\left|\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}}\right| \leqq \frac{e^{-t}}{r} + \frac{e^{-\epsilon t}}{1-e^{-r}} < \left(\frac{1}{r} + \frac{1}{1-e^{-r}}\right)e^{-\epsilon t},$$

so that finally, for $t \geqq 0$ and $\Re(s) \geqq \epsilon$, $|\,s\,| \leqq A$

$$\left|\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}}\right| < M(\epsilon, A)\cdot e^{-\epsilon t}.$$

Consequently the integral

$$\int_0^\infty\left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}}\right)dt$$

converges uniformly in respect to s for $\Re(s) \geqq \epsilon$, $|\,s\,| \leqq A$, and

$$\left|\int_0^\infty\left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}}\right)e^{-nt}dt\right| < M(\epsilon, A)\int_0^\infty e^{-(\epsilon+n)t}dt = \frac{M(\epsilon, A)}{\epsilon+n} \to 0$$

as $n \to \infty$, so that finally

$$\psi(s) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1 - e^{-t}} \right) dt, \qquad \Re(s) > 0. \tag{89}$$

We may transform this integral by writing $t = \log (1 + u$, $\Delta = e^\delta - 1$,

$$\psi(s) = \lim_{\delta \to 0} \int_\delta^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1 - e^{-t}} \right) dt = \lim_{\delta \to 0} \left(\int_\delta^\infty \frac{e^{-t}}{t} dt - \int_\Delta^\infty \frac{du}{u(1+u)^s} \right)$$

and since

$$0 < \int_\delta^\Delta \frac{e^{-t}}{t} dt < \int_\delta^\Delta \frac{dt}{t} = \log \frac{e^\delta - 1}{\delta} \to 0 \text{ as } \delta \to 0,$$

this may be written

$$\psi(s) = \lim_{\delta \to 0} \left(\int_\Delta^\infty \frac{e^{-t}}{t} dt - \int_\Delta^\infty \frac{du}{u(1+u)^s} \right),$$

or finally, replacing u by t in the last integral,

$$\psi(s) = \int_0^\infty (e^{-t} - (1+t)^{-s}) \frac{dt}{t}, \qquad \Re(s) > 0. \tag{90}$$

Again, replacing e^{-t} by t in (89), we obtain

$$\psi(s) = \int_0^1 \left(- \frac{1}{\log t} - \frac{t^{s-1}}{1 - t} \right) dt, \qquad \Re(s) > 0. \tag{91}$$

For $s = 1$, we have $\psi(s) = - C$, whence by (89), (90), (91)

$$\begin{aligned} C &= \int_0^\infty \left(\frac{1}{1 - e^{-t}} - \frac{1}{t} \right) e^{-t} dt \\ &= \int_0^\infty \left(\frac{1}{1+t} - e^{-t} \right) \frac{dt}{t} \\ &= \int_0^1 \left(\frac{1}{1-t} + \frac{1}{\log t} \right) dt, \end{aligned} \tag{92}$$

and consequently

$$\begin{aligned} \psi(s) + C &= \int_0^\infty \frac{e^{-t} - e^{-st}}{1 - e^{-t}} dt \\ &= \int_0^\infty ((1+t)^{-1} - (1+t)^{-s}) \frac{dt}{t}, \qquad \Re(s) > 0 \\ &= \int_0^1 \frac{1 - t^{s-1}}{1 - t} dt. \end{aligned} \tag{93}$$

Since (89) converges uniformly for $\Re(s) \geqq \epsilon$, $| s | \leqq A$, the corresponding integral for $\log \Gamma(s)$ may be obtained by integration according to theor.

XX, but it is equally simple to start from (J 3″) for $a = 1$, or rather the equivalent formula

$$\log \Gamma(s) = \lim_{n \to \infty} \left[(s-1) \log (n+1) - \sum_{\nu=1}^{n} \log \frac{s+\nu-1}{\nu} \right].$$

For $\Re(s) > 0$, (87) gives

$$\log \frac{s+\nu-1}{\nu} = \int_0^\infty \frac{e^{-\nu t} - e^{-(s+\nu-1)t}}{t} dt, \qquad \log (n+1) = \int_0^\infty \frac{e^{-t} - e^{-(n+1)t}}{t} dt,$$

$$\log \Gamma(s) = \lim_{n \to \infty} \int_0^\infty \left[(s-1)(e^{-t} - e^{-(n+1)t}) - \sum_{\nu=1}^{n} (e^{-\nu t} - e^{-(s+\nu-1)t}) \right] \frac{dt}{t}$$

$$= \lim_{n \to \infty} \int_0^\infty \left[(s-1)e^{-t} + \frac{e^{-st} - e^{-t}}{1 - e^{-t}} \right] (1 - e^{-nt}) \frac{dt}{t}.$$

When $\Re(s) \geqq \epsilon$ ($\epsilon < 1$), $|s| \leqq A$, it is seen exactly as before that

$$\left| \left[(s-1)e^{-t} + \frac{e^{-st} - e^{-t}}{1 - e^{-t}} \right] \frac{1}{t} \right| < M(\epsilon, A) e^{-\epsilon t}$$

for $t \geqq 0$, where M depends only on ϵ and A but not on s, and that

$$(94) \qquad \log \Gamma(s) = \int_0^\infty \left[(s-1)e^{-t} + \frac{e^{-st} - e^{-t}}{1 - e^{-t}} \right] \frac{dt}{t}, \qquad \Re(s) > 0,$$

the integral being uniformly convergent for $\Re(s) \geqq \epsilon$, $|s| \leqq A$. In the same manner as (90) was derived from (89), we also find

$$(95) \quad \log \Gamma(s) = \int_0^\infty \left[(s-1)e^{-t} + \frac{(1+t)^{-s} - (1+t)^{-1}}{\log (1+t)} \right] \frac{dt}{t}, \quad \Re(s) > 0.^{18}$$

14. **Integrals of Raabe and Binet.** The integral of Raabe[19]

$$(96) \qquad \int_0^1 \log \Gamma(s+a) da = s \log s - s + \log \sqrt{2\pi}, \qquad (s \text{ not on cut}),$$

results most readily from (76), which gives upon integration

$$\int_0^1 \log \Gamma(s+a) da = s \log s - s + \log \sqrt{2\pi} + \int_0^1 da \int_0^\infty \frac{P_1(t-a) dt}{s+t}.$$

Here the double integral vanishes, being the limit as $n \to \infty$ of

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[18] Eq. (89) and (91) are due to Gauss (J^{11}), (90) to Dirichlet, P. G. L., Sur les intégrales Eulériennes, Journ. f. Math., vol. 15 (1836), pp. 258–263 = Werke, vol. 1, pp. 271–278. The last of (93) was given by Legendre, A. M., Exercices de calcul intégral, vol. 2 (1817), p. 45, and (94) and (95) by Cauchy, A. L., Exercices d'analyse et de physique mathématique, vol. 2 (Paris, 1841), p. 380.

[19] Raabe, J. L., Angenäherte Bestimmung der Factorenfolge, etc., Journ. f. Math., vol. 25 (1843), pp. 146–159.

$$\int_0^1 da \int_0^n \frac{P_1(t-a)dt}{s+t} = \int_0^n \frac{dt}{s+t} \int_0^1 P_1(t-a)da,$$

and by (41), $\int_0^1 P_1(t-a)da = P_2(t-1) - P_2(t) = 0$. Another proof may be derived by means of (J 3′):

$$\begin{aligned}
\int_0^1 \log \Gamma(s+a)da &= \lim_{n\to\infty} \int_0^1 \left[\log \Gamma(n) + (s+a)\log n - \sum_{\nu=0}^{n-1} \log(s+a+\nu)\right] da \\
&= \lim_{n\to\infty} \left[\log \Gamma(n) + (s+\tfrac{1}{2})\log n - \int_0^n \log(s+a)da\right] \\
&= \lim_{n\to\infty} [\log \Gamma(n) + (s+\tfrac{1}{2})\log n - (s+n)\log(s+n) \\
&\qquad\qquad + s\log s + n] \\
&= s\log s - s + \lim_{n\to\infty} [\log \Gamma(n) - (n-\tfrac{1}{2})\log n + n],
\end{aligned}$$

and the last limit is log $\sqrt{2\pi}$, as is seen from Stirling's formula. We shall now derive an expression for $\omega(s, a)$ similar to the integral for log $\Gamma(s)$ in (94). From (76) we obtain, since $P_1(x)$ has the period unity,

$$\omega(s, a) = \lim_{n\to\infty} \int_0^n \frac{P_1(u-a)}{s+u} du = \lim_{n\to\infty} \sum_{\nu=0}^{n-1} \int_0^1 \frac{P_1(u-a)}{s+u+\nu} du,$$

and from (88), for $\Re(s) > 0$,

$$\frac{1}{s+u+\nu} = \int_0^\infty e^{-(s+u+\nu)t} dt,$$

so that

$$\omega(s, a) = \lim_{n\to\infty} \sum_{\nu=0}^{n-1} \int_0^1 du \int_0^\infty P_1(u-a) e^{-(s+u+\nu)t} dt.$$

By theorem XX, the order of integration in the repeated integrals may be inverted, and since $P_1(x) = -\frac{1}{2} - x$ for $-1 < x < 0$ but $P_1(x) = \frac{1}{2} - x$ for $0 < x < 1$, we have

$$\begin{aligned}
\int_0^1 P_1(u-a)e^{-ut}du &= \int_0^a (a - \tfrac{1}{2} - u)e^{-tu}du + \int_a^1 (a + \tfrac{1}{2} - u)e^{-ut}du \\
&= \frac{1}{t}\left[\left(a - \tfrac{1}{2} - \frac{1}{t}\right)(1 - e^{-t}) + e^{-at}\right],
\end{aligned}$$

as is readily seen upon integration by parts, so that

$$\begin{aligned}
\omega(s, a) &= \lim_{n\to\infty} \sum_{\nu=0}^{n-1} \int_0^\infty \left[\left(a - \tfrac{1}{2} - \frac{1}{t}\right)(1 - e^{-t}) + e^{-at}\right] e^{-(s+\nu)t} \frac{dt}{t} \\
&= \lim_{n\to\infty} \int_0^\infty \left[a - \tfrac{1}{2} - \frac{1}{t} + \frac{e^{-at}}{1 - e^{-t}}\right] (1 - e^{-nt}) e^{-t} \frac{dt}{t}.
\end{aligned}$$

Exactly as in the proof of (89), we see that for $t \geqq 0$,

$$\left| \left[a - \tfrac{1}{2} - \frac{1}{t} + \frac{e^{-at}}{1 - e^{-t}} \right] \cdot \frac{1}{t} \right| < M$$

and that consequently, for $\mathfrak{R}(s) > 0$, the limiting process may be performed under the integral sign, so that finally[20]

$$(97) \quad \omega(s, a) = \int_0^\infty \left(a - \tfrac{1}{2} - \frac{1}{t} + \frac{e^{-at}}{1 - e^{-t}} \right) \frac{e^{-st} dt}{t}, \qquad \mathfrak{R}(s) > 0.$$

Writing

$$(98) \qquad f(t, a) = \left(a - \tfrac{1}{2} - \frac{1}{t} + \frac{e^{-at}}{1 - e^{-t}} \right) \cdot \frac{1}{t},$$

this function is holomorphic in the neighborhood of $t = 0$, and consequently $\frac{\partial^k f(t, a)}{\partial t^k}$ is bounded in this neighborhood. Using Leibniz's formula for obtaining this derivative, each term will contain a negative power of t as factor, and consequently $\frac{\partial^k f(t, a)}{\partial t^k} \to 0$ as $t \to \infty$. We therefore have

$$\left| \frac{\partial^k f(t, a)}{\partial t^k} \right| < M_k \text{ for } t \geqq 0,$$

where M_k is independent of t. Now integrating by parts in (97), we obtain

$$\omega(s, a) = \left[- \sum_{k=1}^{2m-2} \frac{\partial^{k-1} f(t, a)}{\partial t^{k-1}} \frac{e^{-st}}{s^k} \right]_{t=0}^{t=\infty} + \frac{1}{s^{2m-2}} \int_0^\infty \frac{\partial^{2m-2} f(t, a)}{\partial t^{2m-2}} e^{-st} dt$$

and since $\mathfrak{R}(s) = |s| \cos \theta > 0$ and the partial derivatives of $f(t, a)$ are bounded, each term in the sum vanishes at the upper limit and

$$\left| \int_0^\infty \frac{\partial^{2m-2} f(t, a)}{\partial t^{2m-2}} e^{-st} dt \right| < \int_0^\infty M_{2n-2} e^{-|s| \cos \theta \cdot t} dt = \frac{M_{2m-2}}{|s| \cos \theta},$$

we see that

$$(99) \quad \omega(s, a) = \sum_{k=1}^{2m-1} \left[\frac{\partial^{k-1} f(t, a)}{\partial t^{k-1}} \right]_{t=0} \cdot \frac{1}{s^k} + \frac{M_{2m-2} \cdot h}{|s|^{2m-1} \cos \theta}, \qquad |h| < 1$$

Consequently (99) is an asymptotic expansion of $\omega(s, a)$ for $\mathfrak{R}(s) > 0$, and must therefore coincide with (78), whence

$$\left[\frac{\partial^{k-1} f(t, a)}{\partial t^{k-1}} \right]_{t=0} = (-1)^{[k(k-1)]/2} (k-1)! P_{k+1}(a), \qquad k = 1, 2, \cdots.$$

Using this in the expansion of $f(t, a)$ by Taylor's theorem, we obtain a formula which is essentially identical with (49).

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[20] For $a = 0$, this expression is due to Binet (J^{24}).

It should be noted that this method of deriving the asymptotic expansion of $\omega(s, a)$ only shows the expansion to be valid in the half-plane $\Re(s) > 0$, whereas the method of § 12 is valid when s is not on the negative real axis.

15. **The generalized Schaar integral.** To obtain this integral, we shall replace each term in the expression (J 24)

$$\frac{d\psi(s)}{ds} = \sum_{\nu=0}^{\infty} \frac{1}{(s+\nu)^2}$$

by an integral over a rational function (instead of the exponential used in § 13). Let $s = x + yi$ and $x > 0$; then

$$\begin{aligned}\int_0^\infty \frac{sdt}{t^2+s^2} &= \int_0^\infty \frac{i}{2}\left(\frac{1}{t-y+xi} - \frac{1}{t+y-xi}\right)dt \\ &= \frac{i}{2}\int_0^\infty \left(\frac{t-y-xi}{(t-y)^2+x^2} - \frac{t+y+xi}{(t+y)^2+x^2}\right)dt \\ &= \frac{i}{2}\left[\frac{1}{2}\log\frac{(t-y)^2+x^2}{(t+y)^2+x^2} - i\arctan\frac{t-y}{x} - i\arctan\frac{t+y}{x}\right]_{t=0}^{t=\infty} \\ &= \frac{\pi}{2}.\end{aligned}$$

Now let $\sigma = \xi + \eta i$ and $\xi > 0$, then

$$\frac{\pi}{2}(s-\sigma) = \int_0^\infty \left(\frac{s^2}{t^2+s^2} - \frac{\sigma^2}{t^2+\sigma^2}\right)dt,$$

whence

$$\frac{1}{s+\sigma} = \frac{2}{\pi}\int_0^\infty \frac{t^2dt}{(t^2+s^2)(t^2+\sigma^2)}, \qquad \Re(s) > 0, \qquad \Re(\sigma) > 0,$$

and this is also true for $\sigma = 0$, since the integral then reduces to the preceding one. Differentiating in respect to x, we obtain

$$(100) \qquad \frac{1}{(s+\sigma)^2} = \frac{2}{\pi}\int_0^\infty \frac{2st^2dt}{(t^2+s^2)^2(t^2+\sigma^2)}, \qquad \Re(s) > 0, \qquad \Re(\sigma) > 0,$$

and this is also true for $\sigma = 0$. In fact we have [21]

$$(101) \qquad |t^2+s^2| \geqq (t^2+|s|^2)\cos\theta \text{ for } t \geqq 0, \qquad s = |s|e^{\theta i}, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

and for $s = |s|e^{\theta i}$, $\sigma = |\sigma|e^{\phi i}$, $-\frac{\pi}{2} + \epsilon \leqq \theta$, $\varphi \leqq \frac{\pi}{2} - \epsilon$, $A \geqq |s| \geqq \delta$, $|\sigma| \geqq \delta$, the integrand in (100) is consequently less than

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[21] Since $|t^2+s^2|^2 = |t^2+|s|^2e^{2\theta i}|^2 = (t^2+|s|^2\cos 2\theta)^2 + (|s|^2\sin 2\theta)^2 = (t^2+|s|^2)^2\cos^2\theta + (t^2-|s|^2)^2\sin^2\theta$.

$$\frac{2At^2}{\sin^3 \epsilon (t^2 + \delta^2)^3},$$

so that the integral (100) is uniformly convergent in respect to s in the region considered, and theorem XXI is applicable; a similar proof applies to the case $\sigma = 0$. In order to obtain a formula applying to all values of s except those on the negative real axis, let us assume $-\pi + 2\epsilon \leqq \theta \leqq \pi - 2\epsilon$; then we may determine an angle α such that $-\frac{\pi}{2} + \epsilon \leqq \alpha \leqq \frac{\pi}{2} - \epsilon$ and $-\frac{\pi}{2} + \epsilon \leqq \theta - \alpha \leqq \frac{\pi}{2} - \epsilon$. In (100) now replace s by $se^{-\alpha i}$ and σ by $\nu e^{-\alpha i}$, where ν is a positive integer or zero, and form the sum from $\nu = 0$ to $\nu = n - 1$; we obtain

$$\sum_{\nu=0}^{n-1} \frac{1}{(s+\nu)^2} = \frac{2}{\pi} \int_0^\infty \frac{2se^{3\alpha i}t^2}{(t^2e^{2\alpha i} + s^2)^2} \sum_{\nu=0}^{n-1} \frac{1}{t^2e^{2\alpha i} + \nu^2}\, dt.$$

But by (101), $| t^2e^{2\alpha i} + \nu^2 | \geqq (t^2 + \nu^2) \cos \alpha \geqq \nu^2 \sin \epsilon$, and for $| s | \geqq \delta$,

$$(102) \qquad \begin{aligned} | t^2e^{2\alpha i} + s^2 | = | t^2 + s^2 e^{-2\alpha i} | &\geqq (t^2 + | s |^2) \cos (\theta - \alpha) \\ &\geqq (t^2 + \delta^2) \sin \epsilon; \end{aligned}$$

consequently

$$\left| \int_0^\infty \frac{2se^{3\alpha i}t^2}{(t^2e^{2\alpha i} + s^2)^2} \sum_{\nu=n}^{\infty} \frac{1}{t^2e^{2\alpha i} + \nu^2}\, dt \right| \leqq \frac{2 | s |}{\sin^3 \epsilon} \sum_{\nu=n}^{\infty} \frac{1}{\nu^2} \cdot \int_0^\infty \frac{t^2 dt}{(t^2 + \delta^2)^2} \to 0$$

as $n \to \infty$, since the last integral converges and $\Sigma \frac{1}{\nu^2}$ is a convergent series; therefore

$$\psi'(s) = \sum_0^\infty \frac{1}{(s+\nu)^2} = \frac{2}{\pi} \int_0^\infty \frac{2se^{3\alpha i}t^2}{(t^2e^{2\alpha i} + s^2)^2} \sum_{\nu=0}^{\infty} \frac{1}{t^2e^{2\alpha i} + \nu^2}\, dt.$$

But from the partial fraction expansion of the cotangent it follows (see § 8) that

$$\sum_{\nu=0}^{\infty} \frac{2t}{t^2 + \nu^2} = \frac{2\pi}{e^{2\pi t} - 1} + \frac{1}{t} + \pi,$$

and changing t into $te^{\alpha i}$, we find

$$\psi'(s) = \frac{2}{\pi} \int_0^\infty \frac{se^{\alpha i} dt}{(t^2e^{2\alpha i} + s^2)^2} + 2 \int_0^\infty \frac{se^{2\alpha i} t dt}{(t^2e^{2\alpha i} + s^2)^2} + 4 \int_0^\infty \frac{se^{2\alpha i} t}{(t^2e^{2\alpha i} + s^2)^2} \frac{dt}{e^{2\pi t e^{\alpha i}} - 1}.$$

The first of these integrals is found, by (100) for $\sigma = 0$, to equal $1/2s^2$, the value $1/s$ of the second is obtained by direct integration, and the third converges uniformly for $A \geqq | s | \geqq \delta$, as is seen from (100) and the fact that

$$\left|\frac{t}{e^{2\pi t e^{\alpha i}}-1}\right| < \begin{cases} \dfrac{1}{\sin\epsilon} & \text{for } 0 < t < 1, \\ \dfrac{1}{t^2 \sin^3 \epsilon} & \text{for } t > 1.^{22} \end{cases} \tag{103}$$

Hence the application of theorem XXI to the third integral gives

$$\psi'(s) = \frac{1}{s} + \frac{1}{2s^2} - 2\frac{\partial}{\partial x}\int_0^\infty \frac{e^{2\alpha i}}{t^2 e^{2\alpha i} + s^2}\cdot\frac{t\,dt}{e^{2\pi t e^{\alpha i}} - 1},$$

and integration in respect to x,

$$\psi(s) = \log s - \frac{1}{2s} + c(y) - 2\int_0^\infty \frac{e^{2\alpha i}}{t^2 e^{2\alpha i} + s^2}\cdot\frac{t\,dt}{e^{2\pi t e^{\alpha i}} - 1},$$

where we have denoted the integration constant by $c(y)$. Now in $s = x + yi$ let $x \to \infty$; then $\theta = \arctan y/x \to 0$, so that for any α where $|\alpha| \leqq \pi/2 - \epsilon$ and α has the same sign as y, we have $|\theta - \alpha| \leqq \pi/2 - \epsilon$ for x sufficiently large. Then, by (102) and (103)

$$\left|\int_0^\infty \frac{e^{2\alpha i}}{t^2 e^{2\alpha i} + s^2}\cdot\frac{t\,dt}{e^{2\pi t e^{\alpha i}} - 1}\right| < \frac{1}{|s|^2 \sin\epsilon}\left(\int_0^1 \frac{dt}{\sin\epsilon} + \int_1^\infty \frac{dt}{t^2 \sin^3\epsilon}\right) \to 0$$

as $x \to \infty$, and since also $\psi(s) - \log s \to 0$ by (84), it follows that $c(y) = 0$ for all values of y. Consequently

$$\psi(s) = \log s - \frac{1}{2s} - 2\int_0^\infty \frac{e^{2\alpha i}}{t^2 e^{2\alpha i} + s^2}\cdot\frac{t\,dt}{e^{2\pi t e^{\alpha i}} - 1}, \tag{104}$$
$$|\theta| < \pi, \qquad |\theta| < \frac{\pi}{2}, \qquad |\theta - \alpha| < \frac{\pi}{2},$$

and upon integration by parts

$$\psi(s) = \log s - \frac{1}{2s} + \frac{1}{\pi}\int_0^\infty \frac{e^{\alpha i}(s^2 - t^2 e^{2\alpha i})}{(t^2 e^{2\alpha i} + s^2)^2}\log(1 - e^{-2\pi t e^{\alpha i}})\,dt. \tag{105}$$

Since $|\alpha| < \pi/2$, we have $|e^{-2\pi t e^{\alpha i}}| = e^{-2\pi t \cos\alpha} < 1$ for $t > 0$, and consequently

$$|\log(1 - e^{-2\pi t e^{\alpha i}})| = \left|\sum_{\nu=1}^{\infty}\frac{e^{-2\pi t e^{\alpha i}}}{\nu}\right| \leqq \sum_{\nu=1}^{\infty}\frac{e^{-2\pi t\cos\alpha}}{\nu} \tag{106}$$
$$= -\log(1 - e^{-2\pi t\cos\alpha});$$

the last two expressions show that $\log(1 - e^{-2\pi t e^{\alpha i}})$ becomes infinite as $\log t$ for $t \to 0$ and vanishes as $e^{-2\pi t\cos\alpha}$ when $t \to \infty$. Together with

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[22] Since $|e^{2\pi t e^{\alpha i}} - 1| \geqq |e^{2\pi t e^{\alpha i}}| - 1 = e^{2\pi t\cos\alpha} - 1 \geqq e^{2\pi t\sin\epsilon} - 1 \geqq$ either of $2\pi t\sin\epsilon$ or $\dfrac{(2\pi t\sin\epsilon)^3}{3!}$.

(102), this shows that the integral in (105) converges uniformly in respect to s in the region previously considered, and we may therefore integrate (105) in respect to x, obtaining

$$\log \Gamma(s) = (s - \tfrac{1}{2}) \log s - s + c(y) - \frac{1}{\pi}\int_0^\infty \frac{se^{\alpha i}}{t^2e^{2\alpha i} + s^2} \log (1 - e^{-2\pi t e^{\alpha i}})dt$$

From (102) and the preceding remarks on the logarithm it follows that the last integral approaches zero as $x \to \infty$, and from (76) and (78) $\log \Gamma(s) - (s - \frac{1}{2}) + s \to \log \sqrt{2\pi}$, so that $c(y)$ equals the latter constant and we finally obtain

$$\begin{aligned} \log \Gamma(s) &= (s - \tfrac{1}{2}) \log s - s + \log \sqrt{2\pi} \\ &\qquad - \frac{1}{\pi}\int_0^\infty \frac{se^{\alpha i}}{t^2e^{2\alpha i} + s^2} \log (1 - e^{-2\pi t e^{\alpha i}})dt, \end{aligned} \tag{107}$$

$$| \theta | < \pi, \qquad | \alpha | < \frac{\pi}{2}, \qquad | \theta - \alpha | < \frac{\pi}{2},$$

where, by (76), the last integral equals $\omega(s, 0)$.[23]

Making $s = 1$ and $\alpha = 0$ in (104) and (105), we obtain two expressions[24] for Euler's constant

$$\begin{aligned} C &= \tfrac{1}{2} + 2\int_0^\infty \frac{1}{t^2 + 1} \frac{tdt}{e^{2\pi t} - 1} \\ &= \tfrac{1}{2} - \pi\int_0^\infty \frac{1 - t^2}{(t^2 + 1)^2} \log (1 - e^{-2\pi t})dt. \end{aligned} \tag{108}$$

To obtain the asymptotic expansion of $\omega(s, 0)$ from (107), we expand the first factor under the integral sign in negative powers of s:

$$\frac{se^{\alpha i}}{t^2e^{2\alpha i} + s^2} = \sum_{k=1}^{m-i} \frac{(-1)^{k-1}e^{(4k-3)\alpha i}t^{2k-2}}{s^{2k-1}} + \frac{(-1)^{m-1}e^{(4m-3)\alpha i}t^{2m-2}}{s^{2m-3}(t^2e^{2\alpha i} + s^2)}$$

and obtain

$$\omega(s, 0) = \sum_{k=1}^{m-1} \frac{(-1)^k}{s^{2k-1}} \cdot \frac{1}{\pi}\int_0^\infty e^{(4k-3)\alpha i}t^{2k-2} \log (1 - e^{-2\pi t e^{\alpha i}})dt + R_m,$$

$$R_m = \frac{(-1)^m}{s^{2m-3}} \cdot \frac{1}{\pi}\int_0^\infty \frac{e^{(4m-3)\alpha i}t^{2m-2}}{t^2e^{2\alpha i} + s^2} \log (1 - e^{-2\pi t e^{\alpha i}})dt.$$

From (102) and (106) it follows that

$$| R_m | \leqq \frac{1}{| s |^{2m-3}} \cdot \frac{1}{\pi}\int_0^\infty \frac{t^{2m-2}}{| s |^2 \cos (\theta - \alpha)} \cdot - \log (1 - e^{-2\pi t \cos \alpha})dt,$$

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[23] For $\alpha = 0$, $\Re(s) > 0$, (107) is due to Schaar, Mémoire sur les intégrales eulériennes et sur la convergence d'une certaine classe de séries, Mém. Ac. Belgique, vol. 22 (1848), pp. 3–25.

[24] Poisson, S. D., Mémoire sur les intégrales définies, Journ. de l'École Polytechn., cahier 18 (1813), p. 305.

or introducing $t\cos\alpha$ as integration variable

$$|R_m| \leqq \frac{1}{|s|^{2m-1}\cos^{2m-1}\alpha\cos(\theta-\alpha)}\cdot\frac{1}{\pi}\int_0^\infty - t^{2m-2}\log(1-e^{-2\pi t})dt.$$

Hence $|s|^{2m-2}|R_m| \to 0$ as $|s| \to \infty$ and the expansion above is asymptotic, so that it must coincide with (79) for $a = 0$. Expressing $P_{2k}(0)$ in terms of B_k and identifying coefficients, we therefore obtain

$$(109)\qquad \frac{B_k}{(2k-1)\cdot 2k} = -\frac{1}{\pi}\int_0^\infty e^{(4k-3)\alpha i}t^{2k-2}\log(1-e^{-2\pi t e^{\alpha i}})dt;$$

making $k = m$, $\alpha = 0$ and introducing in the last expression for $|R_m|$, we find

$$(110)\qquad |R_m| \leqq \frac{B_m}{(2m-1)\cdot 2m}\cdot\frac{1}{\cos^{2m-1}\alpha\cos(\theta-\alpha)}\cdot\frac{1}{|s|^{2m-1}}.$$

By choosing α for a given θ so as to make the second denominator a maximum, this form of the remainder will frequently be much smaller than the one in (79).[25]

16. **Sonin's form of the remainder in Stirling's formula.** In the remainder term in $\omega(s, 0)$ obtained at the end of the preceding paragraph, we now assume s real and positive and make $\alpha = 0$, whence

$$(-1)^{m-1}R_m = \frac{1}{s^{2m-3}}\cdot\frac{1}{\pi}\int_0^\infty \frac{t^{2m-2}}{t^2+s^2}\log\frac{1}{1-e^{-2\pi t}}dt,$$

or writing $t = su$,

$$(-1)^{m-1}R_m = \frac{1}{\pi}\int_0^\infty u^{2m-2}\cdot\log\frac{1}{1-e^{-2\pi s u}}\cdot\frac{du}{1+u^2}.$$

For $s > 0$, $u > 0$, $0 < e^{-2\pi s u} < 1$, and

$$\log\frac{1}{1-e^{-2\pi s u}} = \sum_1^\infty \frac{e^{-2n\pi s u}}{n}$$

$$= \sum_1^\infty \frac{e^{-2n\pi(s+\lambda)u}}{n}\cdot e^{2n\pi\lambda u};$$

assuming $\lambda > 0$, $e^{2n\pi\lambda u}$ is positive and increases with n, whence

$$\log\frac{1}{1-e^{-2\pi s u}} > e^{2\pi\lambda u}\cdot\sum_1^\infty\frac{e^{-2n\pi(s+\lambda)u}}{n}$$

$$= e^{2\pi\lambda u}\log\frac{1}{1-e^{-2\pi(s+\lambda)u}}.$$

Now assume λ so large that for every $u \geqq 0$

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[25] For further details on this point, compare Lindelöf, Calcul des résidus, pp. 98–102.

$$e^{2\pi\lambda u} \geqq 1 + u^2, \tag{111}$$

then the last expression for R_m gives

$$\frac{1}{\pi}\int_0^\infty u^{2m-2}\log\frac{1}{1-e^{-2\pi s u}}\,du > (-1)^{m-1}R_m$$
$$> \frac{1}{\pi}\int_0^\infty u^{2m-2}\log\frac{1}{1-e^{-2\pi(s+\lambda)u}}\,du,$$

and introducing the new integration variables su in the first and $(s+\lambda)u$ in the second of these integrals

$$\frac{1}{s^{2m-1}}\cdot\frac{1}{\pi}\int_0^\infty t^{2m-2}\log\frac{1}{1-e^{-2\pi t}}\,dt > (-1)^{m-1}R_m$$
$$> \frac{1}{(s+\lambda)^{2m-1}}\cdot\frac{1}{\pi}\int_0^\infty t^{2m-2}\log\frac{1}{1-e^{-2\pi t}}\,dt.$$

Making $k = m$ and $\alpha = 0$ in (109), it follows that

$$\frac{B_m}{(2m-1)\cdot 2m}\cdot\frac{1}{s^{2m-1}} > (-1)^{m-1}R_m > \frac{B_m}{(2m-1)\cdot 2m}\cdot\frac{1}{(s+\lambda)2^{m-1}}.$$

To secure as close a limitation of R_m as possible, we have to find the smallest value of λ which will satisfy (111). It is clear that for this smallest value of λ, the equality sign in (111) must obtain for one or more values of u besides $u = 0$, and consequently the smallest admissible value of λ equals the maximum of

$$\lambda = \frac{1}{2\pi}\cdot\frac{1}{u}\log(1+u^2)$$

This maximum must satisfy the equation

$$\frac{d(2\pi\lambda)}{du} = -\frac{1}{u^2}\log(1+u^2) + \frac{2}{1+u^2} = 0;$$

let

$$f(u^2) = u^2\frac{d(2\pi\lambda)}{du} = \frac{2u^2}{1+u^2} - \log(1+u^2),$$

so that

$$f'(u^2) = \frac{1-u^2}{(1+u^2)^2},$$

and consequently $f(u^2)$ increases for $u < 1$ and decreases for $u > 1$. But for u small, $f(u^2) = u^2 + \cdots > 0$, so that finally $f(u^2) = 0$ can have only one root, for which $u > 1$. The numerical calculation of this root gives $1 + u^2 = 4.92154\cdots$ and the corresponding value of λ is $0.12808\cdots$, so that we may now replace (80) by the following:

$$(112)\quad \omega(s,0)=\sum_{k=1}^{m-1}\frac{(-1)^{k-1}B_k}{(2k-1)\cdot 2k}\cdot\frac{1}{s^{2k-1}}+\frac{(-1)^{m-1}B_m}{(2m-1)\cdot 2m}\,\frac{1}{(s+h)^{2m-1}},$$

$$0<h<0.12808\cdots,\qquad s>0.$$

17. **Kummer's and Lerch's trigonometric series.** The function $\dfrac{\Gamma(s)}{\Gamma(1-s)}$ is holomorphic for $0<s<1$, and becomes infinite as $\log 1/s$ when $s\rightarrow 0$ and as log $(1-s)$ as $s\rightarrow 1$, so that $\displaystyle\int_0^1\left|\log\frac{\Gamma(s)}{\Gamma(1-s)}\right|ds$ exists. Consequently, this function is expansible in a Fourier series (35) uniformly convergent for $\epsilon\leqq s\leqq 1-\epsilon$, where the coefficients are given by (34):

$$a_n=2\int_0^1\log\frac{\Gamma(s)}{\Gamma(1-s)}\cdot\cos 2n\pi s ds,\qquad b_n=2\int_0^1\log\frac{\Gamma(s)}{\Gamma(1-s)}\sin 2n\pi s ds,$$

and replacing s by $1-s$ in the expression for a_n, it is seen that

$$a_n=2\int_0^1\log\frac{\Gamma(1-s)}{\Gamma(s)}\cos 2n\pi s ds=-a_n$$

or $a_n=0$. To calculate b_n, we observe that, by (J 3′),

$$\frac{\Gamma(s)}{\Gamma(1-s)}=\lim_{m\rightarrow\infty}m^{2s-1}\cdot\frac{(1-s)(2-s)\cdots(m-s)}{s(1+s)\cdots(m-1+s)},$$

so that

$$\tfrac{1}{2}b_n=\lim_{m\rightarrow\infty}\Bigg[\log m\cdot\int_0^1(2s-1)\sin 2n\pi s ds+\sum_{\nu=1}^{m}\int_0^1\log(\nu-s)\sin 2n\pi s ds$$
$$+\sum_{\nu=0}^{m-1}\int_0^1\log(\nu+s)\sin 2n\pi s ds\Bigg].$$

Integrating by parts,

$$\int_0^1(2s-1)\sin 2n\pi s ds=-\frac{1}{n\pi},$$

and taking $\nu-s$ and $\nu+s$ respectively as new integration variables

$$\sum_{\nu=1}^{m}\int_0^1\log(\nu-s)\sin 2n\pi s ds=-\sum_{\nu=1}^{m}\int_{\nu-1}^{\nu}\log s\sin 2n\pi s ds$$
$$=-\int_0^m\log s\sin 2n\pi s ds,$$

$$\sum_{\nu=0}^{m-1}\int_0^1\log(\nu+s)\sin 2n\pi s ds=\sum_{\nu=0}^{m-1}\int_{\nu}^{\nu+1}\log s\sin 2n\pi s ds$$
$$=\int_0^m\log s\sin 2n\pi s ds,$$

whence

$$\tfrac{1}{2}b_n = \lim_{m\to\infty}\left[-\frac{\log m}{n\pi} - 2\int_0^m \log s \sin 2n\pi s ds\right],$$

and integrating the last integral by parts

$$\tfrac{1}{2}b_n = \frac{1}{n\pi}\lim_{m\to\infty}\left[\int_0^m \frac{1-\cos 2n\pi s}{s}ds - \log m\right]$$

$$= \frac{1}{n\pi}\lim_{m\to\infty}\left[\int_0^{1/2} \frac{1-\cos 4mn\pi s}{s}ds - \log mn + \log n\right],$$

so that finally, replacing mn by m in the limit,

$$\tfrac{1}{2}b_n = \frac{1}{n\pi}(\log n + K),$$

where

$$K = \lim_{m\to\infty}\left[\int_0^{1/2} \frac{1-\cos 4m\pi s}{s}ds - \log m\right].$$

To calculate K, we shall replace the integral by one which may be evaluated in finite terms. Observing that

$$\varphi(s) = \frac{1}{s} - \frac{\pi}{\sin \pi s}$$

is holomorphic for $0 \leqq s \leqq \tfrac{1}{2}$, we may write

$$\int_0^{1/2} \frac{1-\cos 4m\pi s}{s}ds - \pi\int_0^{1/2} \frac{1-\cos 4m\pi s}{\sin \pi s}ds$$

$$= \int_0^{1/2} \varphi(s)ds - \int_0^{1/2} \varphi(s)\cos 4m\pi s ds.$$

Now

$$\int_0^{1/2} \varphi(s)ds = \left[\log \frac{s}{\tan \frac{\pi}{2} s}\right]_{s=0}^{s=1/2} = \log \tfrac{1}{2} + \log \frac{\pi}{2} = \log 2\pi - 3 \log 2,$$

and integrating by parts

$$\int_0^{1/2} \varphi(s)\cos 4m\pi s ds = \frac{1}{4m\pi}\Big[\varphi(s)\sin 4m\pi s\Big]_{s=0}^{s=1/2}$$

$$- \frac{1}{4m\pi}\int_0^{1/2} \varphi'(s)\sin 4m\pi s ds$$

$$\to 0 \text{ as } m \to \infty,$$

since $\varphi(s)$ and $\varphi'(s)$ are holomorphic, and therefore bounded, in the interval of integration. Consequently

$$K = \lim_{m\to\infty}\left[\pi\int_0^{1/2}\frac{1-\cos 4m\pi s}{\sin \pi s}\,ds - \log m\right] + \log 2\pi - 3\log 2,$$

and since we have

$$\frac{1-\cos 4m\pi s}{\sin \pi s} = 2\sum_{\nu=1}^{2m}\sin (2\nu - 1)\pi s,$$

the integral is readily evaluated, giving

$$K = \lim_{m\to\infty}\left[\sum_{\nu=1}^{2m}\frac{2}{2\nu-1} - \log m\right] + \log 2\pi - 3\log 2.$$

By the definition of Euler's constant,

$$\sum_{\nu=1}^{4m}\frac{2}{\nu} - 2\log 4m \to 2C, \qquad \sum_{\nu=1}^{2m}\frac{1}{\nu} - \log 2m \to C$$

as $m \to \infty$, and by subtraction,

$$\sum_{\nu=1}^{2m}\frac{2}{2\nu-1} - \log m - 3\log 2 \to C,$$

so that finally

$$K = C + \log 2\pi,$$

and consequently

$$\tfrac{1}{2}\log\frac{\Gamma(s)}{\Gamma(1-s)} = \sum_{1}^{\infty}\frac{\log n + C + \log 2\pi}{n\pi}\sin 2n\pi s, \qquad 0 < s < 1.$$

But by (40) we have, for $0 < s < 1$

$$\sum_{1}^{\infty}\frac{C + \log 2\pi}{n\pi}\sin 2n\pi s = (C + \log 2\pi)(\tfrac{1}{2} - s),$$

and furthermore, by (J 6),

$$\tfrac{1}{2}\log\frac{\Gamma(s)}{\Gamma(1-s)} = \tfrac{1}{2}\log\frac{\Gamma(s)^2}{\Gamma(s)\Gamma(1-s)} = \log\Gamma(s) + \tfrac{1}{2}\log\frac{\sin \pi s}{\pi},$$

whence we obtain Kummer's series[26]

$$\log\Gamma(s) + \tfrac{1}{2}\log\frac{\sin\pi s}{\pi} + (C + \log 2\pi)(s - \tfrac{1}{2}) = \sum_{1}^{\infty}\frac{\log n}{n\pi}\sin 2n\pi s, \tag{113}$$

$$0 < s < 1,$$

the series being uniformly convergent for $\epsilon \leqq s \leqq 1 - \epsilon$.

The differentiation term by term of the series (113) gives a divergent series; a trigonometric expansion of $\psi(s)$ is however furnished by the following theorem:[27]

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[26] Kummer, E. E., Beitrag zur Theorie der Function $\Gamma(x)$, Journ. f. Math., vol. 35 (1847), pp. 1–4.

Let

$$f(x) = \sum_{n=1}^{\infty} \frac{c_n}{n} \sin 2n\pi x$$

be convergent for $0 < x < 1$; then the derivative of $f(x)$ is given in this interval by

$$f'(x) \cdot \frac{\sin \pi x}{\pi} = \sum_{n=0}^{\infty} (c_n - c_{n+1}) \sin (2n + 1)\pi x,$$

where $c_0 = 0$, provided that the last series converges uniformly for $\epsilon \leqq x \leqq 1 - \epsilon$, where ϵ is as small as we please.

Writing

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} (c_n - c_{n+1}) \sin (2n + 1)\pi x \\ &= \sum_{n=0}^{N} (c_n - c_{n+1}) \sin (2n + 1)\pi x + R_N, \\ &= \sum_{n=1}^{N} c_n \sin (2n + 1)\pi x - \sum_{n=1}^{N+1} c_n \sin (2n - 1)\pi x + R_N \\ &= \sum_{n=1}^{N} c_n \cdot 2 \sin \pi x \cos 2n\pi x - c_{N+1} \sin (2N + 1)\pi x + R_N, \end{aligned}$$

where, on account of the uniform convergence, $| R_N |$ may be made as small as we please for $\epsilon \leqq x \leqq 1 - \epsilon$ by taking N sufficiently large, we have

$$\frac{\pi}{\sin \pi x} g(x) = \sum_{n=1}^{N} c_n \cdot 2\pi \cos 2n\pi x - \pi c_{N+1} \frac{\sin (2N + 1)\pi x}{\sin \pi x} + \frac{\pi R_N}{\sin \pi x},$$

and integrating between the limits ϵ and x

$$\int_{\epsilon}^{x} \frac{\pi}{\sin \pi x} g(x) dx = \sum_{n=1}^{N} \frac{c_n}{n} (\sin 2n\pi x - \sin 2n\pi\epsilon)$$
$$- c_{N+1} \int_{\epsilon}^{x} \frac{\pi \sin (2N + 1)\pi x}{\sin \pi x} dx + \int_{\epsilon}^{x} \frac{\pi R_N}{\sin \pi x} dx.$$

Integrating by parts,

$$c_{N+1} \int_{\epsilon}^{x} \frac{\pi \sin (2N + 1)\pi x}{\sin \pi x} dx = \frac{{}_{N+1}}{2N + 1} \left[\frac{\cos (2N + 1)\pi\epsilon}{\sin \pi\epsilon} \right.$$
$$\left. - \frac{\cos (2N + 1)\pi x}{\sin \pi x} - \int_{\epsilon}^{x} \frac{\pi \cos (2N + 1)\pi x \cos \pi x}{\sin^2 \pi x} dx \right]$$

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[27] Lerch, M., Sur la différentiation d'une classe de séries trigonométriques, Ann. Ec. Norm., ser. 3, vol. 12 (1895), pp. 351–361.

and here the expression in brackets is evidently bounded for $\epsilon \leqq x \leqq 1 - \epsilon$ and all N, so that our integral vanishes as $N \to \infty$, since then $\frac{c_{N+1}}{2N+1} \to 0$[28].

Furthermore,

$$\left| \int_\epsilon^x \frac{\pi R_N}{\sin \pi x} dx \right| < \frac{\pi}{\sin \pi \epsilon} \int_\epsilon^x | R_N | \, dx \to 0 \text{ as } N \to \infty$$

on account of the uniform convergence of the series $g(x)$, and we therefore have, letting $N \to \infty$,

$$\int_\epsilon^x \frac{\pi}{\sin \pi x} g(x) dx = \sum_{n=1}^{\infty} \frac{c_n}{n} (\sin 2n\pi x - \sin 2n\pi\epsilon) = f(x) - f(\epsilon),$$

whence our proposition follows by differentiation.

In the special case $c_n = \log n/\pi$,

$$g(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \log \frac{n}{n+1} \cdot \sin (2n+1)\pi x,$$

and since $\log \frac{n+1}{n}$ is positive and decreases as n increases, we have by (38)

$$\left| \sum_{\nu=n}^{n+p} \log \frac{\nu+1}{\nu} \sin (2\nu + 1)\pi x \right| \leqq \frac{1}{\sin \pi\epsilon} \cdot \log \frac{n+1}{n},$$

so that the series $g(x)$ converges uniformly for $\epsilon \leqq x \leqq 1 - \epsilon$. We may therefore apply Lerch's theorem to (113) and obtain Lerch's series (l. c.)

$$\begin{aligned} &\psi(s) \sin \pi s + \frac{\pi}{2} \cos \pi s + (C + \log 2\pi) \sin \pi s \\ &\qquad = \sum_{n=1}^{\infty} \log \frac{n}{n+1} \cdot \sin (2n+1)\pi s, \qquad 0 < s < 1, \end{aligned} \tag{114}$$

the series being uniformly convergent for $\epsilon \leqq s \leqq 1 - \epsilon$. It should be noted that this series is not a Fourier series according to the definition in § 7.

CHAPTER III.

The Gamma function defined as a definite integral.

18. The Euler integrals of the first and second kind and their elementary properties. In the preceding chapter, the various integrals connected with the Gamma function were derived from the definition of $\Gamma(s)$ as Euler's infinite product. As we shall now proceed to show, the same results may also be obtained by defining $\Gamma(s)$ by means of Euler's integral of the second kind (61)

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[28] This is an immediate consequence of the convergence of the series for $f(x)$ in (0, 1) and the following theorem, due to Cantor: When $\sum_0^\infty (a_n \cos nx + b_n \sin nx)$ converges in an interval (a, b), then $a_n \to 0$, $b_n \to 0$ as $n \to \infty$. We omit the proof, since we shall apply Lerch's theorem only to the case $c_n = \log n/\pi$, where it is evident that $c_{n+1}/(2n+1) \to 0$ as $n \to \infty$.

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt, \qquad \Re(s) > 0. \tag{115}$$

As shown in § 10, the integral converges uniformly for $\epsilon \leqq \Re(s) \leqq A$, where ϵ is as small and A as large as we please, but does not converge for $\Re(s) < 0$. Since therefore s *must be restricted to the half-plane* $\Re(s) > 0$ *throughout the present chapter*, certain formulas (for instance, the asymptotic expansions in § 12, which were there proved for all s not on the negative real axis) will have a larger region of validity than that shown by the proofs to be given in the following. Integrating by parts, we find

$$\int_0^\infty e^{-t}t^s dt = \Big[-e^{-t}t^s\Big]_{t=0}^{t=\infty} + s\int_0^\infty e^{-t}t^{s-1}dt = s\int_0^\infty e^{-t}t^{s-1}dt,$$

or

$$\Gamma(s+1) = s\Gamma(s), \tag{116}$$

which is one of the fundamental properties of the Gamma function ($J4a$). By direct integration, (61) gives $\Gamma(1) = 1$, whence from (116), for a positive integral n, $\Gamma(n) = (n-1)!$ The formulas (62) and (63)

$$\frac{\Gamma(s)}{a^s} = \int_0^\infty e^{-at}t^{s-1}dt, \qquad a > 0, \tag{117}$$

$$\frac{d\Gamma(s)}{ds} = \int_0^\infty e^{-t}t^{s-1}\log t\,dt \tag{118}$$

are derived from (115) exactly as in § 10.

Euler's integral of the first kind is defined as

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, \qquad \Re(\alpha) > 0, \qquad \Re(\beta) > 0;$$

introducing the new integration variable $u = \dfrac{t}{1-t}$, whence $t = \dfrac{u}{1+u}$, we find

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}\,du. \tag{119}$$

To express this integral in terms of the Gamma function we write $a = 1 + u$, $s = \alpha + \beta$ in (117):

$$\int_0^\infty e^{-(1+u)t}t^{\alpha+\beta-1}dt = \frac{\Gamma(\alpha+\beta)}{(1+u)^{\alpha+\beta}},$$

multiply by $u^{\alpha-1}du$ and integrate from 0 to ∞, which gives

$$\int_0^\infty du \int_0^\infty e^{-(1+u)t}u^{\alpha-1}t^{\alpha+\beta-1}dt = \Gamma(\alpha+\beta)\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du.$$

If we replace the integrand to the left by its absolute value (which amounts to replacing α and β by their real parts), the new repeated integral converges, since it equals the right hand member with $\Re(\alpha)$ and $\Re(\beta)$ substituted for α and β; by theorem XXII, we may therefore reverse the order of integration in the repeated integral and obtain

$$\int_0^\infty dt \int_0^\infty e^{-(1+u)t}u^{\alpha-1}t^{\alpha+\beta-1}\,du = \Gamma(\alpha+\beta)\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du.$$

But by (117)

$$\int_0^\infty e^{-ut}u^{\alpha-1}du = \frac{\Gamma(\alpha)}{t^\alpha},$$

and consequently

$$\Gamma(\alpha)\int_0^\infty e^{-t}t^{\beta-1}dt = \Gamma(\alpha+\beta)\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du,$$

or finally, using (115)

$$(120)\qquad \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \Re(\alpha) > 0, \qquad \Re(\beta) > 0,$$

which by (119), is equivalent to (71).

To prove that

$$(121)\qquad \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \qquad 0 < \Re(s) < 1,$$

(which is J 6 with a restriction on $\Re(s)$ imposed by the method of the present chapter), we make $\alpha = s$, $\beta = 1 - s$ in (117), so that

$$\Gamma(\alpha+\beta) = \Gamma(1) = 1,$$

and show that

$$(122)\qquad \int_0^\infty \frac{u^{s-1}}{1+u}du = \frac{\pi}{\sin \pi s}, \qquad 0 < \Re(s) < 1.$$

The integral to the left converges uniformly for $\epsilon \leqq \Re(s) \leqq 1 - \epsilon$ by theorem XIX, since $|u^{s-1}/(1+u)| \leqq u^{\epsilon-1}$ for $0 < u < 1$ and $|u^{s-1}/(1+u)| \leqq u^{-\epsilon-1}$ for $u > 1$, and the integrals

$$\int_0^1 u^{\epsilon-1}du, \qquad \int_1^\infty u^{-\epsilon-1}du$$

both converge. By theorem XXIII, our integral therefore defines a function of s which is holomorphic for $0 < \Re(s) < 1$, and the expression to the right in (121) is also holomorphic in the same region. To show that both sides in (121) are equal throughout this region, it is consequently sufficient to prove their equality for an infinity of values of s having at least one limiting value interior to the region, for instance the values $s = (2p + 1)/2q$, where p and q are any positive integers such that $p < q$. Making $u = t^{2q}$ in (119), it is therefore sufficient to show that

$$\int_{-\infty}^\infty \frac{t^{2p}}{1 + t^{2q}}\, dt = 2\int_0^\infty \frac{t^{2p}}{1 + t^{2q}}\, dt = \frac{\pi}{q \sin \dfrac{2p+1}{2q}\pi}.$$

The roots of $1 + t^{2q} = 0$ are $t_n = e^{(2n+1)\pi i/2q}$ and $\bar{t}_n = e^{-(2n+1)\pi i/2q}$ $(n = 0, 1, \cdots, q-1)$, and decomposing into partial fractions, we have

$$\frac{t^{2p}}{1 + t^{2q}} = \sum_{n=0}^{q-1} \left(\frac{A_n}{t - t_n} + \frac{\bar{A}_n}{t - \bar{t}_n} \right),$$

where

$$A_n = \frac{t_n^{2p}}{2qt_n^{2q-1}} = -\frac{1}{2q} t_n^{2p+1}, \qquad \bar{A}_n = -\frac{1}{2q}\bar{t}_n^{2p+1}.$$

Now $t_n\bar{t}_n = 1$,

$$t_n + \bar{t}_n = 2\cos\frac{2n+1}{2q}\pi$$

is real, and

$$\frac{t_n - \bar{t}_n}{2i} = \sin\frac{2n+1}{2q}\pi > 0;$$

hence

$$\frac{A_n}{t - t_n} + \frac{\bar{A}_n}{t - \bar{t}_n} = \frac{(A_n + \bar{A}_n)t - (A_n\bar{t}_n + \bar{A}_n t_n)}{t^2 - (t_n + \bar{t}_n)t + 1}$$

$$= \frac{1}{2}\frac{(A_n + \bar{A}_n)(2t - t_n - \bar{t}_n)}{t^2 - (t_n + \bar{t}_n)t + 1}$$

$$+ \frac{1}{2}\frac{(A_n - \bar{A}_n)(t_n - \bar{t}_n)}{\left(t - \dfrac{t_n + \bar{t}_n}{2}\right)^2 + \left(\dfrac{t_n - \bar{t}_n}{2i}\right)^2}$$

and

$$\lim_{t\to\infty}\int_{-t}^{t}\left(\frac{A_n}{t-t_n}+\frac{\bar{A}_n}{t-\bar{t}_n}\right)dt = \tfrac{1}{2}(A_n+\bar{A}_n)\lim_{t\to\infty}\log\frac{t^2-(t_n+\bar{t}_n)t+1}{t^2+(t_n+\bar{t}_n)t+1}$$

$$+\tfrac{1}{2}(A_n-\bar{A}_n)(t_n-\bar{t}_n)\cdot\frac{1}{\dfrac{t_n-\bar{t}_n}{2i}}$$

$$\times\left[\arctan\frac{t-\dfrac{t_n+\bar{t}_n}{2}}{\dfrac{t_n-\bar{t}_n}{2i}}\right]_{t=-\infty}^{t=\infty}$$

$$= \pi i(A_n-\bar{A}_n) = 2\pi\Re(iA_n)$$

$$= \frac{\pi}{q}\Re(-ie^{(2n+1)(2p+1)\pi i/2q}),$$

so that finally

$$\int_{-\infty}^{\infty}\frac{t^{2p}}{1+t^{2q}}dt = \frac{\pi}{q}\Re\left(\sum_{n=0}^{q-1}-ie^{(2n+1)(2p+1)\pi i/2q}\right)$$

$$= \frac{\pi}{q}\Re\left(-ie^{(2p+1)\pi i/2q}\cdot\frac{e^{2q(2p+1)\pi i/2q}-1}{e^{2(2p+1)\pi i/2q}-1}\right)$$

$$= \frac{\pi}{q}\Re\left(-i\cdot\frac{-1-1}{e^{(2p+1)\pi i/2q}-e^{-(2p+1)\pi i/2q}}\right)$$

$$= \frac{\pi}{q}\cdot\frac{1}{\sin\dfrac{2p+1}{2q}\pi},$$

which proves our proposition.[29]

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[29] From (121) we may obtain the expansion in partial fractions of $\pi/\sin\pi s$. For decomposing the integral in two with limits 0, 1 and 1, ∞, and changing u into $1/u$ in the second of these integrals, we obtain

$$\frac{\pi}{\sin\pi s} = \int_0^1\frac{u^{s-1}+u^{-s}}{1+u}du,\qquad 0<\Re(s)<1,$$

and since

$$\frac{1}{1+u} = \sum_{\nu=0}^{n-1}(-1)^\nu u^\nu + (-1)^n\frac{u^n}{1+u},$$

it is seen that

$$\frac{\pi}{\sin\pi s} = \sum_{\nu=0}^{n-1}\frac{(-1)^\nu}{s+\nu}+\sum_{\nu=0}^{n-1}\frac{(-1)^\nu}{\nu+1-s}+(-1)^n\int_0^1\frac{u^{s+n-1}+u^{n-s}}{1+u}du.$$

For $\epsilon \leqq \Re(s) \leqq 1-\epsilon$, we have

$$\left|\int_0^1\frac{u^{s+n-1}+u^{n-s}}{1+u}du\right| < \int_0^1(u^{\epsilon+n-1}+u^{n-(1-\epsilon)})ds = \frac{2}{n+\epsilon}\to 0 \text{ as } n\to\infty,$$

and consequently

$$\frac{\pi}{\sin\pi s} = \sum_0^\infty\frac{(-1)^\nu}{s+\nu}+\sum_0^\infty\frac{(-1)^\nu}{\nu+1-s} = \sum_{-\infty}^{\infty}\frac{(-1)^\nu}{s+\nu},$$

In (120), we now make $\alpha = \beta = s$, obtaining

$$\frac{\Gamma(s)^2}{\Gamma(2s)} = \int_0^1 \frac{u^{s-1}}{(1+u)^{2s}}du + \int_1^\infty \frac{u^{s-1}}{(1+u)^{2s}}du = 2\int_0^1 \frac{u^{s-1}}{(1+u)^{2s}}du,$$

and writing $u = (1 - t^{1/2})/(1 + t^{1/2})$ in the last integral,

$$\frac{\Gamma(s)^2}{\Gamma(2s)} = \frac{1}{2^{2s-1}}\int_0^1 t^{-\frac{1}{2}}(1-t)^{s-1}dt = \frac{1}{2^{2s-1}} \cdot \frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(s+\frac{1}{2})}$$

by (119) and (120). Since (121) gives $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we obtain Legendre's formula (J^{13})

$$\Gamma(s)\Gamma(s+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2s-1}}\Gamma(2s), \qquad \Re(s) > 0. \tag{123}$$

19. Integrals for $\psi(s)$. The infinite product for $\Gamma(s)$. In (118), we replace $\log t$ by the expression

$$\log t = \int_0^\infty \frac{e^{-u} - e^{-tu}}{u}du$$

obtained from (87), and find

$$\Gamma'(s) = \int_0^\infty dt \int_0^\infty e^{-t}t^{s-1}\frac{e^{-u} - e^{-tu}}{u}du = \int_0^\infty dt \int_0^\infty f(t, u)du;$$

since $e^{-u} \gtrless e^{-tu}$ as $t \gtrless 1$, we have

$$|f(t, u)| = \pm\, e^{-t}t^{\Re(s)-1}\frac{e^{-u} - e^{-tu}}{u} \qquad (\pm \text{ as } t \gtrless 1),$$

so that

$$\int_0^\infty dt \int_0^\infty |f(t, u)|\,du = \int_0^1 dt \int_0^\infty |f(t, u)|\,du + \int_1^\infty dt \int_0^\infty |f(t, u)|\,du$$

$$= -\int_0^1 dt \int_0^\infty e^{-t}t^{\Re(s)-1}\frac{e^{-u} - e^{-tu}}{u}du$$

$$+ \int_1^\infty dt \int_0^\infty e^{-t}t^{\Re(s)-1}\frac{e^{-u} - e^{-tu}}{u}du$$

and performing the integrations in respect to u

$$\int_0^\infty dt \int_0^\infty |f(t, u)|\,du = -\int_0^1 e^{-t}t^{\Re(s)-1}\log t\,dt + \int_1^\infty e^{-t}t^{\Re(s)-1}\log t\,dt.$$

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for $0 < \Re(s) < 1$, but since the infinite series converges uniformly in any finite region of the s-plane to which the points $s = 0, \pm 1, \pm 2, \cdots$ are exterior, and consequently the series is a function of s holomorphic in this region, the above equation is valid for *all* values of s distinct from $0, \pm 1, \pm 2, \cdots$. Conversely, we may deduce this partial fraction expansion from that of the cotangent by means of the identity $2/\sin \pi s = \cot \pi s/2 + \cot [\pi(1-s)]/2$, and then the above expansion of the integral gives us a new proof of (121).

Both of the last integrals converge since $\Re(s) > 0$; hence $\int_0^\infty dt \int_0^\infty |f(t, u)|\, du$ converges and by theorem XXII, we may reverse the order of integration in the repeated integral for $\Gamma'(s)$, whence

$$\Gamma'(s) = \int_0^\infty du \int_0^\infty t^{s-1} \frac{e^{-u-t} - e^{-(1+u)t}}{u} dt$$

$$= \int_0^\infty \left[e^{-u}\Gamma(s) - \frac{\Gamma(s)}{(1+u)^s} \right] \frac{du}{u}$$

by (115) and (117), or finally, writing t instead of u in the integral,

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)} = \int_0^\infty (e^{-t} - (1+t)^{-s}) \frac{dt}{t}, \qquad \Re(s) > 0, \tag{124}$$

which is identical with (90). We now define Euler's constant by the relation

$$-C = \psi(1) = \Gamma'(1) = \int_0^\infty (e^{-t} - (1+t)^{-1}) \frac{dt}{t}, \tag{125}$$

whence by subtraction

$$\psi(s) + C = \int_0^\infty ((1+t)^{-1} - (1+t)^{-s}) \frac{dt}{t},$$

and replacing $1 + t$ by e^t

$$\psi(s) + C = \int_0^\infty \frac{e^{-t} - e^{-st}}{1 - e^{-t}} dt. \tag{126}$$

Differentiating under the integral sign,

$$\psi'(s) = \int_0^\infty \frac{te^{-st}}{1 - e^{-t}} dt, \tag{127}$$

where evidently $\left| \frac{1}{1 - e^{-t}} \right| < M$ for $t \geqq 0$, so that for $\Re(s) \geqq \epsilon$

$$\int_0^\infty \left| \frac{te^{-st}}{1 - e^{-t}} \right| dt \leqq \int_0^\infty Mte^{-\epsilon t} dt = \frac{M}{\epsilon^2}.$$

Consequently the integral in (127) converges uniformly for $\Re(s) \geqq \epsilon$, so that the differentiation under the integral sign is legitimate by theorem XXI. We have

$$\frac{te^{-st}}{1 - e^{-t}} = \sum_{\nu=0}^{\infty} te^{-(s+\nu)t}, \qquad \left| \frac{te^{-st}}{1 - e^{-t}} \right| = \frac{te^{-\Re(s)\cdot t}}{1 - e^{-t}} = \sum_{\nu=0}^{\infty} ta^{-(\Re(s)+\nu)t},$$

and since $\int_0^\infty \left| \frac{te^{-st}}{1 - e^{-t}} \right| dt$ converges for $\Re(s) > 0$, theorem XXII cor.

gives

$$\psi'(s) = \sum_{\nu=0}^{\infty} \int_0^{\infty} te^{-(s+\nu)t}dt,$$

or evaluating the integrals by means of (117)

$$\psi'(s) = \sum_{\nu=0}^{\infty} \frac{1}{(s+\nu)^2} = \frac{d}{ds}\sum_{\nu=0}^{\infty}\left(\frac{1}{1+\nu} - \frac{1}{s+\nu}\right)$$

where the last series is uniformly convergent for $\Re(s) \geqq \epsilon$, $|s| \leqq A$, since

$$\left|\frac{1}{1+\nu} - \frac{1}{s+\nu}\right| = \frac{|s-1|}{(1+\nu)\,|s+\nu|} < \frac{A+1}{(1+\nu)(\epsilon+\nu)} < \frac{A+1}{\nu^2}$$

and consequently, remembering that $\psi(1) = -C$,

$$\psi(s) + C = \sum_{\nu=0}^{\infty}\left(\frac{1}{1+\nu} - \frac{1}{s+\nu}\right).$$

This may also be written

$$\frac{d\log\Gamma(s+1)}{ds} + C = \sum_{\nu=1}^{\infty}\left(\frac{1}{\nu} - \frac{1}{s+\nu}\right) = \frac{d}{ds}\sum_{\nu=1}^{\infty}\left(\frac{s}{\nu} - \log\frac{s+\nu}{\nu}\right),$$

the last series being uniformly convergent as before, as is readily seen by expanding the logarithm, and since for $s = 0$, $\Gamma(s+1) = 1$, it follows that

$$\log\Gamma(s+1) + Cs = \sum_{\nu=1}^{\infty}\left(\frac{s}{\nu} - \log\left(1 + \frac{s}{\nu}\right)\right).$$

For $s = 1$, we obtain, since $\Gamma(2) = 1$,

$$C = \sum_{\nu=1}^{\infty}\left(\frac{1}{\nu} - \log\left(1+\frac{1}{\nu}\right)\right)$$
$$= \lim_{n\to\infty}\left(1 + \tfrac{1}{2} + \cdots + \frac{1}{n-1} - \log n\right),$$

which is the definition of Euler's constant given in J § 1, and passing from logarithms to numbers, the preceding equation gives, since $\Gamma(s+1) = s\Gamma(s)$,

$$\frac{1}{\Gamma(s)} = e^{Cs}\cdot s\prod_{\nu=1}^{\infty}\left(1+\frac{s}{\nu}\right)e^{-s/\nu}, \qquad \Re(s) > 0.$$

Except for the condition $\Re(s) > 0$, this is the product formula (J 5).

20. **Integrals for** $\log\Gamma(s)$. **Raabe's integral.** In the same way as in the case of (89), we may show that the integral (126) converges uniformly for $\Re(s) \geqq \epsilon$, $|s| \leqq A$, and consequently

$$(s-1)\psi(1+u(s-1)) + C(s-1) = \int_0^\infty (s-1)\frac{e^{-t} - e^{-t-u(s-1)t}}{1-e^{-t}}\,dt$$

converges uniformly for $\Re(s) \geqq \epsilon$, $|s| \leqq A$, $0 \leqq u \leqq 1$. By theorem XX, the integration in respect to u between the limits 0 and 1 may therefore be performed under the integral sign to the right and gives, since $\Gamma(1) = 1$,

$$\log\Gamma(s) + C(s-1) = \int_0^\infty \frac{(s-1)te^{-t} + e^{-st} - e^{-t}}{1-e^{-t}}\,\frac{dt}{t}.$$

For $s = 2$ we obtain, since $\Gamma(2) = 1$,

$$C = \int_0^\infty \left(\frac{te^{-t}}{1-e^{-t}} - e^{-t}\right)\frac{dt}{t},$$

and multiplying by $(s-1)$ and subtracting from the preceding equation,

$$(128)\quad \log\Gamma(s) = \int_0^\infty \left[(s-1)e^{-t} + \frac{e^{-st} - e^{-t}}{1-e^{-t}}\right]\frac{dt}{t}, \qquad \Re(s) > 0,$$

which is identical with (94) and is seen, as in the case of (89), to converge uniformly for $\Re(s) \geqq \epsilon$, $|s| \leqq A$. We shall now evaluate Raabe's integral

$$\int_0^1 \log\Gamma(s+a)da, \qquad \Re(s) > 0,$$

where, on account of the uniform convergence of (128), $\log\Gamma(s+a)$ is continuous for $\Re(s) > 0$ and $0 \leqq a \leqq 1$. From $\log\Gamma(a) = \log\Gamma(1+a) - \log a$ and the existence of $\int_0^1 \log a\,da$ it therefore follows that $\int_0^1 \log\Gamma(a)da$ exists. The integral

$$I = \int_0^1 \log\Gamma(us+a)da,$$

where $\Re(s) \geqq \epsilon$, $|s| \leqq A$, $\delta \leqq u \leqq 1$ (δ being arbitrarily small) may be differentiated under the integral sign, since the resulting integral

$$\frac{\partial I}{\partial u} = \int_0^1 s\psi(us+a)da$$

converges uniformly for the values of s and u considered. But integrating in respect to a and applying (116), we find

$$\frac{\partial I}{\partial u} = s(\log\Gamma(us+1) - \log\Gamma(us)) = s\log us,$$

whence integrating in respect to u between the limits 0 and 1,

$$\int_0^1 \log \Gamma(s+a)da - \int_0^1 \log \Gamma(a)da = \int_0^1 s \log usdu = s \log s - s.$$

Now, replacing a by $1 - a$,

$$\int_0^1 \log \Gamma(a)da = \int_0^1 \log \Gamma(1-a)da$$

and consequently, applying (121)

$$\int_0^1 \log \Gamma(a)da = \tfrac{1}{2}\int_0^1 (\log \Gamma(a) + \log \Gamma(1-a))da$$

$$= \tfrac{1}{2}\int_0^1 (\log \pi - \log \sin \pi a)da = \log \sqrt{\pi} - \tfrac{1}{2}\int_0^1 \log \sin \pi a da.$$

To the last integral, we apply the formula $\sin \pi a = 2 \sin \frac{\pi a}{2} \sin \frac{\pi(1+a)}{2}$, whence

$$\int_0^1 \log \sin \pi a da = \log 2 + \int_0^1 \log \sin \frac{\pi a}{2} da + \int_0^1 \log \sin \frac{\pi(1+a)}{2} da$$

$$= \log 2 + 2\int_0^{\frac{1}{2}} \log \sin \pi a da + 2\int_0^{\frac{1}{2}} \log \sin \left(\frac{\pi}{2} + \pi a\right) da$$

$$= \log 2 + 2\int_0^{\frac{1}{2}} \log \sin \pi a da + 2\int_{\frac{1}{2}}^1 \log \sin \pi a da$$

$$= \log 2 + 2\int_0^1 \log \sin \pi a da,$$

so that

$$\int_0^1 \log \sin \pi a da = -\log 2$$

and

$$\int_0^1 \log \Gamma(a)da = \log \sqrt{\pi} + \tfrac{1}{2}\log 2 = \log \sqrt{2\pi},$$

and finally

$$\int_0^1 \log \Gamma(s+a)da = s \log s - s + \log \sqrt{2\pi}, \qquad \Re(s) > 0. \tag{129}$$

We may obtain another expression for Raabe's integral by replacing s by $s + a$ in (128):

$$\log \Gamma(s+a) = \int_0^\infty \left[(s+a-1)e^{-t} + \frac{e^{-st-at} - e^{-t}}{1-e^{-t}}\right]\frac{dt}{t}, \tag{130}$$

$$\Re(s) > 0, \qquad 0 \leqq a < 1$$

and integrating in respect to a from 0 to 1; the integral to the right being uniformly convergent in respect to a from $0 \leqq a \leqq 1$, we may perform the integration under the integral sign and obtain

$$(131)\quad \int_0^1 \log \Gamma(s+a)da = \int_0^\infty \left[(s-\tfrac{1}{2})e^{-t} + \frac{e^{-st}}{t} - \frac{e^{-t}}{1-e^{-t}}\right]\frac{dt}{t}, \quad \Re(s) > 0.$$

21. Binet's integral and the asymptotic expansion of $\log \Gamma(s+a)$ and $\psi(s+a)$. From (130) we subtract (131) and find by the aid of (129):

$$\log \Gamma(s+a) - (s \log s - s + \log \sqrt{2\pi})$$
$$= \int_0^\infty \left[(a-\tfrac{1}{2})e^{-t} - \frac{e^{-st}}{t} + \frac{e^{-st-at}}{1-e^{-t}}\right]\frac{dt}{t};$$

from (87) we obtain

$$\log s = \int_0^\infty \frac{e^{-t}-e^{-st}}{t}\,dt,$$

and multiplying by $a - \frac{1}{2}$ and subtracting from the preceding equation,

$$\log \Gamma(s+a) = (s+a-\tfrac{1}{2}) \log s - s + \log \sqrt{2\pi}$$
$$+ \int_0^\infty \left(a - \frac{1}{2} - \frac{1}{t} + \frac{e^{-at}}{1-e^{-t}}\right)\frac{e^{-st}dt}{t}, \qquad \Re(s) > 0,$$

Rewriting equations (97) and (98),

$$(132)\qquad \omega(s, a) = \int_0^\infty \left(a - \frac{1}{2} - \frac{1}{t} + \frac{e^{-at}}{1-e^{-t}}\right)\frac{e^{-st}dt}{t},$$

$$(133)\qquad f(t, a) = \left(a - \frac{1}{2} - \frac{1}{t} + \frac{e^{-at}}{1-e^{-t}}\right)\cdot\frac{1}{t},$$

we find as in § 14, upon successive integration by parts in (132),

$$(134)\quad \omega(s, a) = \left[-\sum_{k=1}^{2m-2} \frac{\partial^{k-1} f(t, a)}{\partial t^{k-1}}\frac{e^{-st}}{s^k}\right]_{t=0}^{t=\infty} + \frac{1}{s^{2m-2}}\int_0^\infty \frac{\partial^{2m-2} f(t, a)}{\partial t^{2m-2}}\,e^{-st}dt.$$

It is now proposed to show that this formula yields the asymptotic expansion (78) for $\Re(s) > 0$, and the proof will be conducted without any comparison of (134) to (77) (since we have not established the latter formula by the methods of the present chapter). To this purpose, we note that $f(t, a)$ is a function of a holomorphic for $0 \leqq a \leqq 1$, and may therefore be expanded in a Fourier series uniformly convergent in this interval

$$f(t, a) = \frac{a_0}{2} + \sum_1^\infty (a_n \cos 2n\pi a + b_n \sin 2n\pi a)$$
$$= \frac{a_0}{2} + \sum_1^\infty (A_n e^{2n\pi ia} + A_{-n} e^{-2n\pi ia}),$$

where, by (34),

$$A_n = \tfrac{1}{2}(a_n - ib_n) = \int_0^1 f(t, a)e^{-2n\pi ia}da, \quad A_{-n} = \tfrac{1}{2}(a_n + ib_n) = \int_0^1 f(t, a)e^{2n\pi ia}da,$$

From (133) we find

$$a_0 = 2\int_0^1 f(t, a)da = 0$$

and

$$\begin{aligned} A_{-n} &= \frac{1}{t}\int_0^1 \left(a - \tfrac{1}{2} - \frac{1}{t} + \frac{e^{-at}}{1 - e^{-t}}\right)e^{2n\pi ia}da \\ &= \frac{1}{t}\left[\frac{1}{2n\pi i}ae^{2n\pi ia}\right]_{a=0}^{a=1} + \frac{1}{t}\left[\frac{1}{2n\pi i - t}\cdot\frac{e^{(2n\pi i - t)a}}{1 - e^{-t}}\right]_{a=0}^{a=1} \\ &= \frac{1}{t}\left(\frac{1}{2n\pi i} - \frac{1}{2n\pi i - t}\right) = -\frac{1}{2n\pi i(2n\pi i - t)}; \end{aligned}$$

A_n is obtained from this by changing the sign of i, so that

$$f(t, a) = -\sum_{n=1}^{\infty}\left(\frac{e^{2n\pi ia}}{2n\pi i(2n\pi i + t)} + \frac{e^{-2n\pi ia}}{2n\pi i(2n\pi i - t)}\right). \tag{135}$$

Differentiating term by term in respect to t, which is legitimate since the series obtained are uniformly convergent for $t \geqq 0$, it is seen that

$$\frac{\partial^{k-1}f(t, a)}{\partial t^{k-1}} = -(k - 1)!\sum_{n=1}^{\infty}\left(\frac{(-1)^{k-1}e^{2n\pi ia}}{2n\pi i(2n\pi i + t)^k} + \frac{e^{-2n\pi ia}}{2n\pi i(2n\pi i - t)^k}\right) \tag{136}$$

and from the uniform convergence for $t \geqq 0$ it follows that $\dfrac{\partial^{k-1}f(t, a)}{\partial t^{k-1}} \to 0$ as $t \to \infty$. For $t = 0$, we obtain

$$\left[\frac{\partial^{k-1}f(t, a)}{\partial t^{k-1}}\right]_{t=0} = -(k - 1)!\sum_{n=1}^{\infty}\frac{(-1)^{k-1}e^{2n\pi ia} + e^{-2n\pi ia}}{(2n\pi i)^{k+1}},$$

and comparing the cases k odd and k even separately to (42), we find

$$\left[\frac{\partial^{k-1}f(t, a)}{\partial t^{k-1}}\right]_{t=0} = (-1)^{k(k-1)/2}(k - 1)!P_{k+1}(a).$$

Since $| 2n\pi i \pm t | \geqq 2n\pi$, it follows from (136) for $k = 2m - 1$ that

$$\left|\frac{\partial^{2m-2}f(t, a)}{\partial t^{2m-2}}\right| < (2m - 2)!\sum_{1}^{\infty}\frac{2}{(2n\pi)^{2m}} = (2m - 2)!P_{2m}(0),$$

and for $s = | s | e^{\theta i}$ $(-\pi/2 < \theta < \pi/2$ since $\Re(s) > 0)$,

$$\left|\int_0^\infty \frac{\partial^{2m-2}f(t, a)}{\partial t^{2m-2}} e^{-st}dt\right| < (2m - 2)!P_{2m}(0) \int_0^\infty e^{-|s|\cos\theta\cdot t}dt$$

$$= (2m - 2)!\frac{P_{2m}(0)}{|s|\cos\theta}.$$

Introducing all this in (134), we finally obtain

$$\begin{gathered}\omega(s, a) = \sum_{k=1}^{2m-2} (-1)^{k(k-1)/2} \frac{(k-1)!P_{k+1}(a)}{s^k} + \frac{(2m-2)!P_{2m}(0)\cdot h}{|s|^{2m-1}\cos\theta},\\ |h| < 1, \qquad 0 \leqq a < 1, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},\end{gathered}\tag{137}$$

which, except for the slightly different form of the remainder term, is identical to (78).

We have

$$\omega^*(s, a) = \log s - \psi(s + a) = (\tfrac{1}{2} - a)\frac{1}{s} - \frac{\partial\omega(s, a)}{\partial s},$$

and by theorem XXI, it is legitimate to differentiate in respect to s under the integral sign in (132), whence

$$\omega^*(s, a) = (\tfrac{1}{2} - a)\frac{1}{s} + \int_0^\infty tf(t, a)e^{-st}dt, \qquad \Re(s) > 0. \tag{138}$$

Integrating by parts, and using (135) in the same manner as above, we find

$$\begin{gathered}\omega^*(s, a) = (\tfrac{1}{2} - a)\frac{1}{s} + \sum_{k=2}^{2m-1} (-1)^{(k-1)(k-2)/2} \frac{(k-1)!P_k(a)}{s^k}\\ \qquad\qquad + \frac{(2m-1)!P_{2m}(0)\cdot h}{|s|^{2m}\cos\theta},\\ |h| < 1, \qquad 0 \leqq a < 1, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},\end{gathered}\tag{139}$$

which differs from (84) only in the form of the remainder term.

22. **The integrals of Schaar and Landsberg.** Let us first assume *s real and positive* and replace $f(t, a)$ in (132) by the series (135), so that

$$\omega(s, a) = -\int_0^\infty \sum_{n=1}^\infty \left(\frac{e^{2n\pi i a}}{2n\pi i(2n\pi i + t)} + \frac{e^{-2n\pi i a}}{2n\pi i(2n\pi i - t)}\right) e^{-st}dt.$$

Since $|2n\pi i \pm t| \geqq 2n\pi$, the absolute value of the general term in the series does not exceed $\left(\frac{1}{(2n\pi)^2} + \frac{1}{(2n\pi)^2}\right)e^{-st}$, and the integral

$$\int_0^\infty \sum_{n=1}^\infty \frac{1}{2\pi^2 n^2} e^{-st} dt = \frac{1}{2\pi^2 s} \sum_1^\infty \frac{1}{n^2}$$

being convergent, the same is the case with

$$\int_0^\infty \sum_{n=1}^\infty \left| \left(\frac{e^{2n\pi ia}}{2n\pi i(2n\pi i + t)} + \frac{e^{-2n\pi ia}}{2n\pi i(2n\pi i - t)} \right) e^{-st} \right| dt.$$

By theorem XXII cor., we may therefore reverse the order of integration and summation in the expression for $\omega(s, a)$, and write

$$\omega(s, a) = - \sum_{n=1}^\infty \int_0^\infty \left(\frac{e^{2n\pi ia}}{2n\pi i(2n\pi i + t)} + \frac{e^{-2n\pi ia}}{2n\pi i(2n\pi i - t)} \right) e^{-st} dt.$$

Replacing t by $2n\pi t/s$ in each integral (this is a real substitution since s is real), we obtain

$$\omega(s, a) = \sum_{n=1}^\infty \int_0^\infty \frac{1}{2n\pi} \left(\frac{e^{-2n\pi ia}}{s + ti} + \frac{e^{2n\pi ia}}{s - ti} \right) e^{-2n\pi t} dt;$$

since $| s \pm ti | \geqq s$, the absolute value of each integrand does not exceed $(1/n\pi s) e^{-2n\pi t}$, and the series

$$\sum_{n=1}^\infty \int_0^\infty \frac{e^{-2n\pi t}}{n\pi s} dt = \frac{1}{2\pi^2 s} \sum_1^\infty \frac{1}{n^2}$$

being convergent, the same is the case with the series

$$\sum_{n=1}^\infty \int_0^\infty \left| \frac{1}{2n\pi} \left(\frac{e^{-2n\pi ia}}{s + ti} + \frac{e^{2n\pi ia}}{s - ti} \right) e^{-2n\pi t} \right| dt.$$

By theorem XXII cor., we may therefore again reverse the order of summation and integration in $\omega(s, a)$, which gives

$$\omega(s, a) = \int_0^\infty \sum_{n=1}^\infty \frac{1}{2n\pi} \left(\frac{e^{-2n\pi(t+ai)}}{s + ti} + \frac{e^{-2n\pi(t-ai)}}{s - ti} \right) dt$$

and summing the infinite series

$$\omega(s, a) = \frac{1}{2\pi} \int_0^\infty \left(\frac{1}{s + ti} \log \frac{1}{1 - e^{-2\pi(t+ai)}} + \frac{1}{s - ti} \log \frac{1}{1 - e^{-2\pi(t-ai)}} \right) dt. \tag{140}$$

Passing to the case of s complex, we may now write for $\Re(s) > 0$

$$\omega(s, a) = \frac{1}{2\pi} \int_0^\infty \frac{1}{s + ti} \log \frac{1}{1 - e^{-2\pi(t+ai)}} dt + \frac{1}{2\pi} \int_0^\infty \frac{1}{s - ti} \log \frac{1}{1 - e^{-2\pi(t-ai)}} dt. \tag{141}$$

In fact, $1/(s + ti)$ and $1/(s - ti)$ are holomorphic in s for $\mathfrak{R}(s) \geqq \epsilon$, $|s| \leqq A$ and $t \geqq 0$; since $|s \pm ti| \geqq \epsilon$ and

$$\left|\log \frac{1}{1 - e^{-2\pi(t \pm ai)}}\right| \leqq \log \frac{1}{1 - e^{-2\pi t}}$$

(which is seen by expansion in series, as in (106)), and furthermore

$$\int_0^\infty \frac{1}{\epsilon} \log \frac{1}{1 - e^{-2\pi t}} dt$$

converges (the integrand becoming infinite as $\log t$ for $t \to 0$ and vanishing as $e^{-2\pi t}$ for $t \to \infty$), both integrals in (141) converge uniformly for $\mathfrak{R}(s) \geqq \epsilon$, $|s| \leqq A$. The application of theorem XXIII now shows that the right side in (141) is a function of s holomorphic for $\mathfrak{R}(s) > 0$; the same being the case with $\omega(s, a)$, and both sides in (141) being equal for s real and positive by (140), it follows that the equality subsists in the entire half-plane $\mathfrak{R}(s) > 0$.

Writing $1 - e^{-2\pi(t \pm ai)} = re^{\pm \theta i}$, we have $r^2 = 1 - e^{-2\pi t} \cos 2\pi a + e^{-4\pi t}$,

$$\theta = \arctan \frac{\sin 2\pi a}{e^{2\pi t} - \cos 2\pi a},$$

and making

$$(142) \qquad \begin{aligned} \psi(t, a) &= \frac{1}{4\pi} \log (1 - e^{-2\pi t} \cos 2\pi a + e^{-4\pi t}), \\ \chi(t, a) &= -\frac{1}{2\pi} \arctan \frac{\sin 2\pi a}{e^{2\pi t} - \cos 2\pi a}, \end{aligned}$$

we find by a simple algebraic transformation of (141)

$$(143) \qquad \omega(s, a) = \int_0^\infty \frac{2t\chi(t, a)}{s^2 + t^2} dt - \int_0^\infty \frac{2s\psi(t, a)}{s^2 + t^2} dt, \qquad \mathfrak{R}(s) > 0.$$

Differentiating (141) in respect to $x = \mathfrak{R}(s)$, we find

$$(144) \qquad \begin{aligned} (\tfrac{1}{2} - a)\frac{1}{s} - \omega^*(s, a) &= \frac{\partial \omega(s, a)}{\partial s} \\ &= -\frac{1}{2\pi} \int_0^\infty \frac{1}{(s + ti)^2} \log \frac{1}{1 - e^{-2\pi(t+ai)}} dt \\ &\quad - \frac{1}{2\pi} \int_0^\infty \frac{1}{(s - ti)^2} \log \frac{1}{1 - e^{-2\pi(t-ai)}} dt; \end{aligned}$$

the differentiation under the integral sign is legitimate by theorem XXI, since the integrals obtained converge uniformly for $\mathfrak{R}(s) \geqq \epsilon$, $|s| \leqq A$, as is seen in exactly the same way as for the corresponding integrals in

(141). Integrating by parts in the formula just obtained, we find for $0 < a < 1$

$$-\frac{1}{2\pi}\int_0^\infty \frac{1}{(s+ti)^2}\log\frac{1}{1-e^{-2\pi(t+ai)}}\,dt = \left[\frac{1}{2\pi i}\cdot\frac{1}{s+ti}\log\frac{1}{1-e^{-2\pi(t+ai)}}\right]_{t=0}^{t=\infty}$$
$$- i\int_0^\infty \frac{1}{s+ti}\cdot\frac{e^{-2\pi(t+ai)}}{1-e^{-2\pi(t+ai)}}\,dt,$$

and the first term to the right equals

$$\frac{1}{2\pi i s}\log(1-e^{-2\pi ai}) = \frac{1}{2\pi i s}\log\left(2e^{\pi(1/2-a)i}\cdot\frac{e^{\pi ai}-e^{-\pi ai}}{2i}\right)$$
$$= \frac{1}{2\pi i s}[\pi(\tfrac{1}{2}-a)i + \log(2\sin\pi a)].$$

The second integral in (144) is found by changing the sign of i, and we finally see that

$$(145)\qquad \begin{aligned}\omega^*(s,a) &= i\int_0^\infty \frac{1}{s+ti}\cdot\frac{e^{-2\pi(t+ai)}}{1-e^{-2\pi(t+ai)}}\,dt \\ &\quad - i\int_0^\infty \frac{1}{s-ti}\cdot\frac{e^{-2\pi(t-ai)}}{1-e^{-2\pi(t-ai)}}\,dt, \quad \Re(s) > 0, \quad 0 < a < 1,\end{aligned}$$

or writing

$$(146)\qquad \begin{aligned}\psi_1(t,a) &= \frac{\cos 2\pi a - e^{-2\pi t}}{e^{2\pi t} - 2\cos 2\pi a + e^{-2\pi t}}, \\ \chi_1(t,a) &= \frac{\sin 2\pi a}{e^{2\pi t} - 2\cos 2\pi a + e^{-2\pi t}},\end{aligned}$$

and transforming (145) algebraically

$$(147)\qquad \omega^*(s,a) = \int_0^\infty \frac{2s\chi_1(t,a)}{s^2+t^2}\,dt + \int_0^\infty \frac{2t\psi_1(t,a)}{s^2+t^2}\,dt, \qquad \Re(s) > 0.$$

(147) is also seen to be true for $a = 0$ by making $a = 0$ in (143), differentiating in respect to x and integrating by parts. In the special cases $a = 0$ and $a = \frac{1}{2}$ these expressions are considerably simplified, and we obtain

$$(148)\qquad \begin{aligned}\omega(s,0) &= \frac{1}{\pi}\int_0^\infty \frac{s}{s^2+t^2}\log\frac{1}{1-e^{-2\pi t}}\,dt, \\ \omega^*(s,0) &= \int_0^\infty \frac{2t}{s^2+t^2}\frac{dt}{e^{2\pi t}-1}, \\ \omega(s,\tfrac{1}{2}) &= \frac{1}{\pi}\int_0^\infty \frac{s}{s^2+t^2}\log\frac{1}{1+e^{-2\pi t}}\,dt, \\ \omega^*(s,\tfrac{1}{2}) &= \int_0^\infty \frac{2t}{s^2+t^2}\frac{dt}{e^{2\pi t}+1}.\end{aligned}$$

The integrals (148) are due to Schaar,[23] and (143) and (147) to Landsberg.[30] Using the identity

$$\frac{1}{s^2+t^2}=\sum_{k=1}^{m-1}\frac{(-1)^{k-1}t^{2k-2}}{s^{2k}}+\frac{(-1)^{m-1}t^{2m-2}}{s^{2m-2}(s^2+t^2)}$$

in (143) and (147), and proceeding in the same manner as at the end of § 15, there result asymptotic expansions which must coincide with (137) and (139). Comparing coefficients, we readily obtain

$$\begin{aligned} P_{2k}(a) &= -\frac{2}{(2k-2)!}\int_0^\infty \psi(t,a)t^{2k-2}dt \\ &= \frac{2}{(2k-1)!}\int_0^\infty \psi_1(t,a)t^{2k-1}dt, \\ P_{2k+1}(a) &= -\frac{2}{(2k-1)!}\int_0^\infty \chi(t,a)t^{2k-1}dt \\ &= \frac{2}{(2k)!}\int_0^\infty \chi_1(t,a)t^{2k}dt. \end{aligned} \tag{149}$$

23. **Kummer's trigonometric series.** Let s be real and $\epsilon \leqq s \leqq 1-\epsilon$; we then obtain from (128)

$$\log\frac{\Gamma(s)}{\Gamma(1-s)}=\int_0^\infty\left[\frac{e^{-st}-e^{-(1-s)t}}{1-e^{-t}}+(2s-1)e^{-t}\right]\frac{dt}{t},$$

and subtracting (125) multiplied by $2s-1$

$$\log\frac{\Gamma(s)}{\Gamma(1-s)}+C(2s-1)=\int_0^\infty\left[\frac{e^{-st}-e^{-(1-s)t}}{1-e^{-t}}+\frac{2s-1}{1+t}\right]\frac{dt}{t}. \tag{150}$$

We now have the Fourier expansion

$$\frac{e^{-st}-e^{-(1-s)t}}{1-e^{-t}}+\frac{2s-1}{1+t}=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos 2n\pi s+b_n\sin 2n\pi s)$$

uniformly convergent for $0 \leqq s \leqq 1$ and a fixed value of t, since the left side is a holomorphic function of s. It is seen from (34) that $a_0=0$ and that for $n>0$

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[30] Landsberg, G., Sur un nouveau développement de la fonction gamma, Mém. Ac. Belgique, vol. 55 (1897).

$$
\begin{aligned}
a_n + ib_n &= 2\int_0^1 \left(\frac{e^{-st} - e^{-(1-s)t}}{1 - e^{-t}} + \frac{2s-1}{1+t}\right) e^{2n\pi i s}ds \\
&= \frac{2}{1-e^{-t}}\left[\frac{e^{(2n\pi i - t)s}}{2n\pi i - t} - \frac{e^{-t+(2n\pi i + t)s}}{2n\pi i + t}\right]_{s=0}^{s=1} + \frac{2}{1+t}\left[\frac{(2s-1)e^{2n\pi i s}}{2n\pi i}\right]_{s=0}^{s=1} \\
&= -\frac{2}{2n\pi i - t} - \frac{2}{2n\pi i + t} - \frac{2i}{n\pi(1+t)} \\
&= \frac{8n\pi i}{4n^2\pi^2 + t^2} - \frac{2i}{n\pi(1+t)},
\end{aligned}
$$

so that

$$
\left[\frac{e^{-st} - e^{-(1-s)t}}{1 - e^{-t}} + \frac{2s-1}{1+t}\right] \cdot \frac{1}{t} = \sum_{n=1}^{\infty} c_n(t)\frac{\sin 2n\pi s}{n\pi}, \tag{151}
$$

$$
0 \leqq s \leqq 1, \qquad t \geqq 0
$$

where

$$
c_n(t) = \frac{1}{t}\left(\frac{8n^2\pi^2}{4n^2\pi^2 + t^2} - \frac{2}{1+t}\right) = \frac{2}{1+t} - \frac{2t}{4n^2\pi^2 + t^2} \tag{152}
$$

increases with n for any fixed positive value of t. We shall now prove that it is permissible to substitute (151) in (150) and integrate term by term, or in other words, that

$$
\int_0^\infty \left(\sum_{\nu=n+1}^{\infty} c_\nu(t)\frac{\sin 2\nu\pi s}{\nu\pi}\right) dt \to 0 \qquad \text{as} \qquad n \to \infty
$$

uniformly for $\epsilon \leqq s \leqq 1 - \epsilon$. Writing

$$
S_\nu = \sum_{\mu=\nu}^{\infty} \frac{\sin 2\mu\pi s}{\pi\mu},
$$

so that

$$
\frac{\sin 2\nu\pi s}{\nu\pi} = S_\nu - S_{\nu+1},
$$

it follows from (38) that for $\epsilon \leqq s \leqq 1 - \epsilon$

$$
| S_\nu | \leqq \frac{1}{\nu\pi \sin \pi\epsilon},
$$

and from the identity of summation by parts

$$
\sum_{\nu=n+1}^{n+p} c_\nu(S_\nu - S_{\nu+1}) = \sum_{\nu=n+1}^{n+p} (c_\nu - c_{\nu-1})S_\nu + c_n S_{n+1} - c_{n+p+1}S_{n+p+1}
$$

we conclude, since $c_\nu(t) - c_{\nu+1}(t) > 0$ for $t > 0$, that

$$\left|\sum_{\nu=n+1}^{n+p} c_\nu(t)\,\frac{\sin 2\nu\pi s}{\nu\pi}\right| \leqq \frac{1}{\pi\sin\pi\epsilon}\left[\sum_{\nu=n+1}^{n+p}(c_\nu(t)-c_{\nu-1}(t))\cdot\frac{1}{\nu} + \frac{|\,c_n(t)\,|}{n+1}+\frac{|\,c_{n+p+1}(t)\,|}{n+p+1}\right]. \tag{153}$$

From (152) we obtain

$$\int_0^\infty c_n(t)dt = \left[\log\frac{(1+t)^2}{4n^2\pi^2+t^2}\right]_{t=0}^{t=\infty} = 2\log 2n\pi, \tag{154}$$

and since $c_n(t) \gtrless 0$ as $t \lessgtr 4n^2\pi^2$,

$$\int_0^\infty |\,c_n(t)\,|\,dt = \int_0^{4n^2\pi^2} c_n(t)dt - \int_{4n^2\pi^2}^\infty c_n(t)dt$$

$$= 2\log\left(1+\frac{1}{4n^2\pi^2}\right) + 2\log 2n\pi < 4\log 2n\pi,$$

so that, by (153)

$$\int_0^\infty\left|\sum_{\nu=n+1}^{n+p} c_\nu(t)\,\frac{\sin 2\nu\pi s}{\nu\pi}\right|dt < \frac{2}{\pi\sin\pi\epsilon}\left[\sum_{\nu=n+1}^{n+p}\frac{1}{\nu}\log\left(1+\frac{1}{\nu-1}\right) + \frac{2\log 2n\pi}{n+1} + \frac{2\log 2(n+p+1)\pi}{n+p+1}\right].$$

Letting $p \to \infty$, and observing that $\log\left(1+\dfrac{1}{\nu-1}\right) < \dfrac{1}{\nu-1}$ and $\displaystyle\sum_{\nu=n+1}^{\infty}\frac{1}{\nu(\nu-1)} = \frac{1}{n}$, we finally see that

$$\left|\int_0^\infty\left(\sum_{\nu=n+1}^{\infty} c_\nu(t)\,\frac{\sin 2\nu\pi s}{\nu\pi}\right)dt\right| < \frac{2}{\pi\sin\pi\epsilon}\left(\frac{1}{n}+\frac{2\log 2n\pi}{n+1}\right) \to 0 \text{ as } n \to \infty$$

uniformly for $\epsilon \leqq s \leqq 1-\epsilon$. Substituting (151) in (150) and integrating term by term with the aid of (154), we find for $0 < s < 1$

$$\log\frac{\Gamma(s)}{\Gamma(1-s)} + C(2s-1) = 2\sum_{n=1}^{\infty}\frac{\log n + \log 2\pi}{n\pi}\sin 2n\pi s,$$

the series being uniformly convergent for $\epsilon \leqq s \leqq 1-\epsilon$. Using (40) and (121) (the latter being identical with J 6), we now immediately obtain Kummer's series (113).

Additional note. Application of complex integration to the theory of the Gamma function. We observed at the beginning of chapter III that when the Gamma function is defined by means of Euler's integral of the second kind, we are thereby restricted to the region of convergence of this integral, viz., the half plane $\mathfrak{R}(s) > 0$. This difficulty is overcome by the use of complex integration and other more advanced parts of the theory of functions of a complex variable. The author's original intention to devote a fourth chapter to an exposition of these methods had to be abandoned through lack of space, and it was found feasible only to give the following brief references to the literature.

A paper by Birkhoff[31] contains what is probably the shortest way of arriving at the main properties of the Gamma function from the function theoretic point of view. Starting from the definition

$$\Gamma(s) = \lim_{n\to\infty} \frac{\varphi(s+n+1)}{s(s+1)\cdots(s+n)},$$

$$\varphi(s) = s^{s-1/2}e^{-s}\sqrt{2\pi}, \qquad |\text{arc } s| < \pi,$$

which is essentially equivalent to (J 15), Birkhoff first proves the formula

$$(J\ 14) \qquad \log \Gamma(s) = (s - \tfrac{1}{2}) \log s - s + \log \sqrt{2\pi} + \omega(s)$$

with the Gudermann expansion of $\omega(s)$ (J 13) and the fundamental property of this function, viz. $|\omega(s)| < K/\rho$, where K is a constant and ρ the distance of s from the nearest point on the negative real axis. Next, the formula $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$ (J 6), the infinite products for $\Gamma(s)$ (J 3 and 5), and Euler's integral of the first kind are established, and after a digression on $\psi(s) = \log s - \omega^*(s)$, $|\omega^*(s)| < K/\rho^2$, Birkhoff's paper closes with the expression of Euler's integral of the first kind in terms of the Gamma function (71 in the present paper).

By the use of Cauchy's integral, a systematic study may be made of a whole class of summation formulas, containing that of Euler, and as an application, the formulas (76) and (82), Binet's integral (97) and those of Schaar (143, 147) may be derived from a common point of view, together with the various asymptotic expansions to which they lead. An excellent exposition of this aspect of the theory is given by Lindelöf.[32]

Attention is finally called to an important paper by Mellin,[33] in which is set forth the intimate connection of the Gamma function with a large class of other transcendental functions (for instance, the Riemann Zeta function). The main purpose of Mellin's paper is however to apply the Gamma function to the systematic solution of a class of linear difference equations (the simplest case of which is solved in J § 11) as well as to a corresponding class of linear differential equations, the coefficients of which are polynomials of the first degree.

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[31] Birkhoff, G. D., Note on the Gamma function, Bulletin Am. Math. Soc., vol. 20 (1913–14), pp. 1–10.

[32] Lindelöf, E., Le calcul des résidus, Paris, Gauthier-Villars, 1905 (Collection Borel).

[33] Mellin, H., Abriss einer einheitlichen Theorie der Gamma- und der hypergeometrischen Funktionen, Math. Annalen, vol. 68 (1910), pp. 305–337.